Examples regarding the applied methodology

Example (SMAA): Let us suppose that we need to rank order alternatives a, b, and c, evaluated based on criteria g_1, g_2, and g_3, as shown in Table 1

Table 1 Evaluations of the alternatives using three criteria

Alternative/Criterion	$g_1(\cdot)$	$g_2(\cdot)$	$g_3(\cdot)$
a_1	10	30	20
a_2	20	10	30
a_3	30	20	10

Table 2 Weight vectors

	$w_1^{(\cdot)}$	$w_2^{(\cdot)}$	$w_3^{(\cdot)}$
w^1	0.2488	0.4210	0.3302
w^2	0.2941	0.4646	0.2413
w^3	0.2496	0.4577	0.2926
w^4	0.4153	0.2325	0.3522
w^5	0.4637	0.3022	0.2342
w^6	0.3095	0.2764	0.4141

Table 3 Comprehensive value of each alternative with respect to the six weight vectors

	$U(\cdot, w^1)$	$U(\cdot, w^2)$	$U(\cdot, w^3)$	$U(\cdot, w^4)$	$U(\cdot, w^5)$	$U(\cdot, w^6)$
a_1	21.7214	21.7047	22.0813	18.1712	18.3851	19.6685
a_2	19.0929	17.7672	18.3491	21.1972	19.3200	21.3768
a_3	19.1857	20.5281	19.5697	20.6317	22.2948	18.9547

As can be seen, the rankings obtained by considering the six weight vectors are different. Indeed, w.r.t. w^1, w^2, and w^3, $a_1 \succ a_3 \succ a_2$ (where $a_1 \succ a_3$ means that a_1 is strictly preferred to a_3); w.r.t. w^4, $a_2 \succ a_3 \succ a_1$, w.r.t. w^5, $a_3 \succ a_2 \succ a_1$, while w.r.t. w^6, $a_2 \succ a_1 \succ a_3$. In this way, we show that the final ranking is strictly dependent on the choice of the weight vectors.

Now, let us show how the indices of SMAA can be computed.
Let us suppose that the whole set of weights W is composed only of $w^1, \dots, w^6$. The rank acceptability indices for the three alternatives related to the three positions are shown in Table 4. As can be seen, $b(a_1, 2) = 16.67\%$ since in one out of the six cases a_1 reached the second position (for the weight vector w^6), while $b(a_3, 2) = 66.67\%$ because a_3 reached the second position in four out of the six cases (for the weights w^1, w^2, w^3, w^4).

Table 4 Rank acceptability indices of the three considered alternatives expressed in percentage terms

	$b(\cdot, 1)$	$b(\cdot, 2)$	$b(\cdot, 3)$
a_1	50.00	16.67	33.33
a_2	33.33	16.67	50.00
a_3	16.67	66.67	16.67

Based on the rank acceptability indices, we can compute the barycenter of the weights giving to each alternative the three different positions shown in table 5.

Table 5 Barycenter of the weights giving to three alternatives one of the positions considered

	$w_1^c(a_1,\cdot)$	$w_2^c(a_1,\cdot)$	$w_3^c(a_1,\cdot)$
$w^c(a_1, 1)$	0.2641	0.4477	0.2880

$\boldsymbol{w^c(a_1, 2)}$	0.3095	0.2764	0.4141
$\boldsymbol{w^c(a_1, 3)}$	0.4395	0.2673	0.2932

	$\boldsymbol{w_1^c(a_2,\cdot)}$	$\boldsymbol{w_2^c(a_2,\cdot)}$	$\boldsymbol{w_3^c(a_2,\cdot)}$
$\boldsymbol{w^c(a_2, 1)}$	0.3624	0.2544	0.3837
$\boldsymbol{w^c(a_2, 2)}$	0.4637	0.3022	0.2342
$\boldsymbol{w^c(a_2, 3)}$	0.2641	0.4477	0.2880

	$\boldsymbol{w_1^c(a_3,\cdot)}$	$\boldsymbol{w_2^c(a_3,\cdot)}$	$\boldsymbol{w_3^c(a_3,\cdot)}$
$\boldsymbol{w^c(a_3, 1)}$	0.4637	0.3022	0.2342
$\boldsymbol{w^c(a_3, 2)}$	0.3019	0.3939	0.3040
$\boldsymbol{w^c(a_3, 3)}$	0.3095	0.2764	0.4141

For example, to compute the weight vector $w^c(a_2, 3)$, that is the average preferences giving to a_2 third position, we can observe from Table 3 that a_2 reaches third position in correspondence with the weight vectors w^1, w^2, w^3; consequently, $W_2^3 = \{w^1, w^2, w^3\}$, and therefore $w^c(a_2, 3)$ is the vector obtained averaging component by component the weight vectors in W_2^3. Finally, in Table 6, we show the pairwise winning indices.

Table 6 Pairwise winning indices expressed in percentage terms

$\boldsymbol{p(\cdot,\cdot)}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$
$\boldsymbol{a_1}$	0	66.67	66.67
$\boldsymbol{a_2}$	33.33	0	16.67
$\boldsymbol{a_3}$	33.33	83.33	0

As can be seen, a_1 is preferred to a_2 and a_3 with the same frequency (66.67%) since it is preferred to both alternatives in correspondence with four out of the six weight vectors considered. In particular, a_1 is preferred to a_2 and a_3 for the weight vectors w^1, w^2, and w^3. In addition to these three weight vectors, a_1 is preferred to a_2 in correspondence with w^4, while a_1 is preferred to a_3 in correspondence with w^6. Analogously, a_2 is preferred to a_3 with a frequency of 16.67%, based on correspondence with only one of the six weight vectors.

As previously shown, the ranking of the alternatives will depend on the choice of the weights assigned to the criteria considered. Therefore, the application of the SMAA methodology permits the drawing of robust conclusions in terms of the frequency of attaining a certain ranking position, as well as in terms of the frequency of preference between alternatives.

Example (SMAA-S: computation of the barycenter): Let us consider four criteria supposing that their ranking is as follows:

$$g_2 \gtrsim g_4 \gtrsim g_1 \gtrsim g_3$$

The corresponding criteria weights have to satisfy the inequalities chain:

$$w_2 \geq w_4 \geq w_1 \geq w_3.$$

The polyedron defined by these inequalities has the vertices:

$$w^{(1)} = (0,1,0,0),\ w^{(2)} = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right), w^{(3)} = \left(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right), w^{(4)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Consequently, the barycenter of the polyedron is $BW = \left(\frac{7}{48}, \frac{25}{48}, \frac{3}{48}, \frac{13}{48}\right)$.

Example (SMAA-S): Let us suppose that the variable V to be explained takes the values $V(a_1)$=35, $V(a_2)$=50, and $V(a_3)$=20, so that $Rank_{Benchmark}$ has a_2, a_1, and a_3 in the first, second, and third ranking position, respectively, that is:

$$Rank_{Benchmark} = \{a_3 \succ a_1 \succ a_3\}$$

There are then six possible rankings of importance for the criteria:

$$P^{(1)} = \{g_1 \gtrsim g_2 \gtrsim g_3\}, P^{(2)} = \{g_1 \gtrsim g_3 \gtrsim g_2\},$$
$$P^{(3)} = \{g_2 \gtrsim g_1 \gtrsim g_3\}, P^{(4)} = \{g_2 \gtrsim g_3 \gtrsim g_1\},$$
$$P^{(5)} = \{g_3 \gtrsim g_1 \gtrsim g_2\}, P^{(6)} = \{g_3 \gtrsim g_2 \gtrsim g_1\}.$$

To the ranking of importance $P^{(1)}$ there is a corresponding set of weight vectors $W^{(1)}$ satisfying the following set of constraints:

$$\left.\begin{array}{c} w_1 \geq w_2 \geq w_3 \\ w_1 + w_2 + w_3 = 1 \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0 \end{array}\right\}$$

In the set of weight vectors $W^{(1)}$ one can find the vector w^1=[1,0,0], as well as w^2=[0.5,0.5,0], and also w^3=[1/3,1/3,1/3]. In $W^{(1)}$ there is an infinity of other weight vectors, such as w^4=[0.4,0.35,0.25], w^5=[0.5,0.3,0.2], and so on. The barycenter of $W^{(1)}$, which can be taken as the weight vector representative of all the weight vectors in $W^{(1)}$, is as follows:

$$BW^{(1)} = \left[\frac{1 + 0.5 + \frac{1}{3}}{3}, \frac{0.5 + \frac{1}{3}}{3}, \frac{\frac{1}{3}}{3}\right] = [0.611, 0.278, 0.111].$$

If we compute the comprehensive value of alternatives a_1, a_2, and a_3 in terms of the weighted sum with respect to weight vector $BW^{(1)}$, we obtain the following:

$$U(a_1,BW^{(1)})=16.667,\ U(a_2,BW^{(1)})=18.333,\ U(a_3,BW^{(1)})=25$$

corresponding to the ranking of alternatives

$$Rank^{(1)} = \{a_3 \succ a_2 \succ a_1\}.$$

Therefore, we obtain a Kendall tau equal to $\tau^{(1)} = 1/3$. Analogously, we obtain:

$$\tau^{(2)} = \frac{1}{3}, \tau^{(3)} = \frac{1}{3}, \tau^{(4)} = \frac{1}{3}, \tau^{(5)} = -1, \tau^{(6)} = -1.$$

Therefore, we derive that the maximal value of the Kendall tau is obtained by $Rank^{(1)}$, $Rank^{(2)}$, $Rank^{(3)}$, and $Rank^{(4)}$ for which in two out of four cases the most important factor, i.e., the one with the greatest weight, is g_1, and in two out of four cases the most important factor is g_2, which suggests that g_1 and g_2 are the most important factors in explaining the variable V.