Dealing with Interaction Between Bipolar Multiple Criteria Preferences in PROMETHEE Methods

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Abstract: In this paper we extend the PROMETHEE methods to the case of interacting criteria on a bipolar scale, introducing the bipolar PROMETHEE method based on the bipolar Choquet integral. In order to elicit parameters compatible with preference information provided by the Decision Maker (DM), we propose to apply the Robust Ordinal Regression (ROR). ROR takes into account simultaneously all the sets of parameters compatible with the preference information provided by the DM considering a necessary and a possible preference relation.

Keywords: PROMETHEE methods, Interaction between criteria, Bipolar Choquet integral.

1 Introduction

In a decision making problem a set of alternatives $A = \{a, b, c, \ldots\}$ is evaluated on a set $G = \{g_1, \ldots, g_n\}$ of evaluation criteria in order to deal with a ranking, choice, or sorting problem (for a survey on Multiple Criteria Decision Analysis (MCDA) see Figueira et al. 2005b). Ranking problems consist into rank ordering all the alternatives from the best to the worst; choice problems consist into selecting a subset of alternatives of A considered good or removing a set of alternatives considered bad while sorting problems consist into assigning each alternative to some predefined and preferentially ordered classes.

To aggregate the evaluations of each alternative on the considered criteria, three methodologies are well known in literature:

- assigning to each alternative a real number $U(a)$ representing the degree of desirability of a on the problem at hand as in the Multiple Attribute Utility Theory (MAUT) (see Keeney and Raiffa 1976);

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- building some outranking preference relation on A as in the case of outranking methods (Brans and Mareschal 2005; Brans and Vincke 1985; Figueira et al. 2005a),

- using a set of "‘if..., then"’ decision rules from the Decision Maker (DM) preference information through Dominance-based Rough Set Approach (DRSA, see Greco et al. 2001, 2005)

Considering for each criterion g_j the set X_j of all possible evaluations of alternatives in A on criterion g_j, in the first model a value function $U : \prod_{j=1}^{n} X_j \rightarrow \mathbb{R}$ is defined so that for each couple of alternatives $a, b \in A$, a is at least as good as b if and only if $U(g_1(a), \ldots, g_n(a)) \geq U(g_1(b), \ldots, g_n(b))$. In its simplest form, this function is additive, i.e., $U(g_1(a), \ldots, g_n(a)) = \sum_{j=1}^{n} u_j(g_j(a))$, where $u_j : X_j \rightarrow \mathbb{R}$ are marginal value functions assigning a value to a representing its evaluation on criterion g_j.

In the third model the aim is to express the relationships between some preferences provided by the DM on alternatives of A and their evaluations on the considered criteria using decision rules such as:

- If the price of a car is lower than 10,000 euros and it consumes at most 1 l of fuel per 20 km, then this car is comprehensively excellent.

The second model will be thoroughly discussed in section 2.

In many decision making problems, alternatives are evaluated with respect to a set of criteria being not mutually preferentially independent (see Wakker 1989). In fact, in most cases, the criteria present a certain form of positive (*synergy*) or negative (*redundancy*) interaction. For example, if one likes sport cars, maximum speed and acceleration are very important criteria. However, since in general speedy cars have also a good acceleration, giving a high weight to both criteria can over evaluate some cars. Thus, it seems reasonable to give maximum speed and acceleration considered together a weight smaller than the sum of the two weights assigned to these criteria when considered separately. In this case we have a redundancy between the criteria of maximum speed and acceleration. On the contrary, we have a synergy effect between maximum speed and price because, in general, speedy cars are also expensive and, therefore, a car which is good on both criteria is very appreciated. In this case, it seems reasonable to give maximum speed and price considered together a weight greater than the sum of the two weights assigned to these criteria when considered separately. In these cases, the aggregation of the evaluations is done by using non-additive integrals the most well known of which are the Choquet integral (Choquet 1953) and the Sugeno integral (Sugeno 1974) (for a comprehensive survey on the use of non-additive integrals in MCDA see Grabisch 1996; Grabisch and Labreuche 2005c, 2010).

In many cases, we have also to take into account that the importance of criteria may also depend on the criteria which are opposed to them. For example, a bad evaluation on aesthetics reduces the importance of

maximum speed. Thus, the weight of maximum speed should be reduced when there is a negative evaluation on aesthetics. In this case, we have an *antagonistic* effect between maximum speed and aesthetics.
Those types of interactions between criteria have been already taken into consideration in the ELECTRE methods (Figueira et al., 2009a). In this paper, we deal with the same problem using the bipolar Choquet integral (Grabisch and Labreuche, 2005a,b) applied to the PROMETHEE I and II methods (Brans and Mareschal, 2005; Brans and Vincke, 1985).

This article extends the short paper published by the authors in Corrente et al. (2012) with respect to which we added the description of the bipolar PROMETHEE I method, the proofs of all theorems presented in Corrente et al. (2012) and a didactic example in which we apply the bipolar PROMETHEE methods and the Robust Ordinal Regression (ROR) (Greco et al., 2008, 2010) being a family of MCDA methods taking into account simultaneously all the sets of preference parameters compatible with the preference information provided by the Decision Maker (DM) using a necessary and a possible preference relation.
The paper is organized as follows. In the next section we recall the basic concepts of the classical PROMETHEE methods; in section 3 we introduce the bipolar PROMETHEE methods; the elicitation of preference information permitting to fix the value of the preference parameters of the model (essentially the bicapacities of the bipolar Choquet integral) is presented in section 4; in the fifth section we apply the ROR to the bipolar PROMETHEE methods; a didactic example is presented in section 6, while the last section provides some conclusions and lines for future research.

2 The classical PROMETHEE methods

PROMETHEE (Brans and Mareschal, 2005; Brans and Vincke, 1985) is a well-known family of MCDA methods, among which the most well known are PROMETHEE I and II, that aggregate preference information of a DM through an outranking relation. Considering for each criterion g_j a weight w_j (representing the importance of criterion g_j within the family of criteria G), an indifference threshold q_j (being the largest difference $d_j(a,b) = g_j(a) - g_j(b)$ compatible with the indifference between alternatives a and b), and a preference threshold p_j (being the minimum difference $d_j(a,b)$ compatible with the preference of a over b), PROMETHEE methods (from now on, when we shall speak of PROMETHEE methods, we shall refer to PROMETHEE I and II) build a non decreasing function $P_j(a,b)$ of $d_j(a,b)$, whose formulation (see Brans and Mareschal 2005 for other formulations) can be stated as follows

$$P_j(a,b) = \begin{cases} 0 & \text{if} \quad d_j(a,b) \leq q_j \\ \frac{d_j(a,b)-q_j}{p_j-q_j} & \text{if} \quad q_j < d_j(a,b) < p_j \\ 1 & \text{if} \quad d_j(a,b) \geq p_j \end{cases}$$

The greater the value of $P_j(a,b)$, the greater the preference of a over b on criterion g_j. For each ordered pair of alternatives $(a,b) \in A \times A$, PROMETHEE methods compute the value $\pi(a,b) = \sum_{j \in \mathcal{J}} w_j P_j(a,b)$ where $\mathcal{J} = \{1, \ldots, n\}$ is the set of indices of criteria in G. $\pi(a,b)$ represents how much alternative a is preferred to alternative b taking into account the whole set of criteria and it can assume values between 0 and 1. Obviously, the greater the value of $\pi(a,b)$, the greater the preference of a over b.

In order to compare an alternative a with all the other alternatives of the set A, PROMETHEE methods compute the negative and the positive flow of a

$$\phi^-(a) = \frac{1}{m-1} \sum_{b \in A \setminus \{a\}} \pi(b,a) \qquad \text{and} \qquad \phi^+(a) = \frac{1}{m-1} \sum_{b \in A \setminus \{a\}} \pi(a,b)$$

where $m = |A|$. These flows represent, on average, how much the alternatives of $A \setminus \{a\}$ are preferred to a and how much a is preferred to the alternatives of $A \setminus \{a\}$. For each alternative $a \in A$, PROMETHEE II computes also the net flow $\phi(a) = \phi^+(a) - \phi^-(a)$. On the basis of the positive and the negative flows, PROMETHEE I provides a partial ranking on A, building a preference ($\mathcal{P}^I$), an indifference ($\mathcal{I}^I$) and an incomparability ($\mathcal{R}^I$) relation. In particular:

$$\begin{cases} a\mathcal{P}^I b \quad \text{iff} & \begin{cases} \Phi^+(a) \geq \Phi^+(b), \\ \Phi^-(a) \leq \Phi^-(b), \\ \Phi^+(a) - \Phi^-(a) > \Phi^+(b) - \Phi^-(b) \end{cases} \\ a\mathcal{I}^I b \quad \text{iff} & \begin{cases} \Phi^+(a) = \Phi^+(b), \\ \Phi^-(a) = \Phi^-(b) \end{cases} \\ a\mathcal{R}^I b \quad \text{otherwise} & \end{cases}$$

On the basis instead of the net flows, the PROMETHEE II method provides a complete ranking on A defining, in a natural way, a preference ($\mathcal{P}^{II}$) and an indifference ($\mathcal{I}^{II}$) relation for which $a\mathcal{P}^{II}b$ iff $\Phi(a) > \Phi(b)$ while $a\mathcal{I}^{II}b$ iff $\Phi(a) = \Phi(b)$.

3 The bipolar PROMETHEE methods

In order to extend the classical PROMETHEE methods to the bipolar framework, we define for each criterion g_j, $j \in \mathcal{J}$, the bipolar preference function $P_j^B : A \times A \rightarrow [-1, 1]$, in the following way:

$$P_j^B(a,b) = P_j(a,b) - P_j(b,a) = \begin{cases} P_j(a,b) & \text{if} \quad P_j(a,b) > 0 \\ -P_j(b,a) & \text{if} \quad P_j(a,b) = 0 \end{cases} \tag{1}$$

It is straightforward proving that $P_j^B(a,b) = -P_j^B(b,a)$ for all $j \in \mathcal{J}$ and for all pairs $(a,b) \in A \times A$.

In this section we propose to aggregate the bipolar vector $P^B(a,b) = \left[P_1^B(a,b), \ldots, P_n^B(a,b)\right]$ through the bipolar Choquet integral.

The bipolar Choquet integral is based on a bicapacity (Grabisch and Labreuche, 2005a,b), being a function $\hat{\mu} : P(\mathcal{J}) \rightarrow [-1,1]$, where $P(\mathcal{J}) = \{(C,D) : C, D \subseteq \mathcal{J} \text{ and } C \cap D = \emptyset\}$, such that

- $\hat{\mu}(\emptyset, \mathcal{J}) = -1, \hat{\mu}(\mathcal{J}, \emptyset) = 1$, $\hat{\mu}(\emptyset, \emptyset) = 0$ (boundary conditions),
- for all $(C,D), (E,F) \in P(\mathcal{J})$, if $C \subseteq E$ and $D \supseteq F$, then $\hat{\mu}(C,D) \leq \hat{\mu}(E,F)$ (monotonicity condition).

According to Greco and Figueira (2003) and Greco et al. (2002), we consider the following expression for a bicapacity $\hat{\mu}$:

$$\hat{\mu}(C,D) = \mu^+(C,D) - \mu^-(C,D), \quad \text{for all } (C,D) \in P(\mathcal{J}) \tag{2}$$

where $\mu^+, \mu^- : P(\mathcal{J}) \rightarrow [0,1]$ such that:

$$\mu^+(\mathcal{J}, \emptyset) = 1, \qquad \mu^+(\emptyset, B) = 0, \ \forall B \subseteq \mathcal{J}, \tag{3}$$

$$\mu^-(\emptyset, \mathcal{J}) = 1, \qquad \mu^-(B, \emptyset) = 0, \ \forall B \subseteq \mathcal{J}, \tag{4}$$

$$\left.\begin{array}{ll} \mu^+(C,D) \leq \mu^+(C \cup \{j\}, D), & \forall (C \cup \{j\}, D) \in P(\mathcal{J}), \ \forall j \in \mathcal{J}, \\ \mu^+(C,D) \geq \mu^+(C, D \cup \{j\}), & \forall (C, D \cup \{j\}) \in P(\mathcal{J}), \ \forall j \in \mathcal{J} \end{array}\right\} \tag{5}$$

$$\left.\begin{array}{ll} \mu^-(C,D) \leq \mu^-(C, D \cup \{j\}), & \forall (C, D \cup \{j\}) \in P(\mathcal{J}), \ \forall j \in \mathcal{J}, \\ \mu^-(C,D) \geq \mu^-(C \cup \{j\}, D), & \forall (C \cup \{j\}, D) \in P(\mathcal{J}), \ \forall j \in \mathcal{J} \end{array}\right\} \tag{6}$$

Let us observe that (5) are equivalent to the constraint

$$\mu^+(C,D) \le \mu^+(E,F), \quad \text{for all} \quad (C,D),(E,F) \in P(\mathcal{J}) \quad \text{such that} \quad C \subseteq E \ \text{ and } \ D \supseteq F,$$

while (6) are equivalent to the constraint

$$\mu^-(C,D) \le \mu^-(E,F), \quad \text{for all} \quad (C,D),(E,F) \in P(\mathcal{J}) \quad \text{such that} \quad C \supseteq E \ \text{ and } \ D \subseteq F.$$

The interpretation of the functions μ^+ and μ^- is the following. Given the pair $(a,b) \in A \times A$, let us consider $(C,D) \in P(\mathcal{J})$ where C is the set of criteria expressing a preference of a over b and D the set of criteria expressing a preference of b over a. In this situation, $\mu^+(C,D)$ represents the importance of criteria from C when criteria from D are opposing them, and $\mu^-(C,D)$ represents the importance of criteria from D opposing C. Consequently, $\hat{\mu}(C,D)$ represents the balance of the importance of C supporting a and D supporting b.

Given $(a,b) \in A \times A$, the bipolar Choquet integral of $P^B(a,b)$ with respect to the bicapacity $\hat{\mu}$ can be written as follows

$$Ch^B(P^B(a,b),\hat{\mu}) = \int_0^1 \hat{\mu}(\{j \in \mathcal{J} : P_j^B(a,b) > t\}, \{j \in \mathcal{J} : P_j^B(a,b) < -t\})dt,$$

while the bipolar comprehensive positive preference of a over b and the comprehensive negative preference of a over b with respect to the bicapacity $\hat{\mu}$ are respectively:

$$Ch^{B+}(P^B(a,b),\hat{\mu}) = \int_0^1 \mu^+(\{j \in \mathcal{J} : P_j^B(a,b) > t\}, \{j \in \mathcal{J} : P_j^B(a,b) < -t\})dt,$$

$$Ch^{B-}(P^B(a,b),\hat{\mu}) = \int_0^1 \mu^-(\{j \in \mathcal{J} : P_j^B(a,b) > t\}, \{j \in \mathcal{J} : P_j^B(a,b) < -t\})dt,$$

where μ^+ and μ^- have been defined before.

From an operational point of view, the bipolar aggregation of $P^B(a,b)$ can be computed as follows: for all the criteria $j \in \mathcal{J}$, the absolute values of these preferences should be re-ordered in a non-decreasing way, as follows: $|P^B_{(1)}(a,b)| \le |P^B_{(2)}(a,b)| \le \ldots \le |P^B_{(j)}(a,b)| \le \ldots \le |P^B_{(n)}(a,b)|$. The bipolar Choquet integral of $P^B(a,b)$ with respect to the bicapacity $\hat{\mu}$ can now be determined:

$$Ch^B(P^B(a,b),\hat{\mu}) = \sum_{j \in \mathcal{J}^>} |P^B_{(j)}(a,b)| \Big[\hat{\mu}\left(C_{(j)}(a,b), D_{(j)}(a,b)\right) - \hat{\mu}\left(C_{(j+1)}(a,b), D_{(j+1)}(a,b)\right)\Big] \tag{7}$$

where $P^B(a,b) = \left[P_j^B(a,b),\ j \in \mathcal{J}\right]$, $\mathcal{J}^> = \{j \in \mathcal{J} \ :\ |P_{(j)}^B(a,b)| > 0\}$, $C_{(j)}(a,b) = \{i \in \mathcal{J}^> \ :\ P_i^B(a,b) \geq |P_{(j)}^B(a,b)|\}$, $D_{(j)}(a,b) = \{i \in \mathcal{J}^> \ :\ -P_i^B(a,b) \geq |P_{(j)}^B(a,b)|\}$ and $C_{(n+1)}(a,b) = D_{(n+1)}(a,b) = \emptyset$.

We give also the following definitions:

$$Ch^{B+}(P^B(a,b),\hat{\mu}) = \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(a,b)| \Big[\mu^+\left(C_{(j)}(a,b), D_{(j)}(a,b)\right) - \mu^+\left(C_{(j+1)}(a,b), D_{(j+1)}(a,b)\right)\Big], \tag{8}$$

$$Ch^{B-}(P^B(a,b),\hat{\mu}) = \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(a,b)| \Big[\mu^-\left(C_{(j)}(a,b), D_{(j)}(a,b)\right) - \mu^-\left(C_{(j+1)}(a,b), D_{(j+1)}(a,b)\right)\Big]. \tag{9}$$

Proposition 3.1. *Given a set of criteria $\mathcal{J}$ and a bicapacity $\hat{\mu}$ defined on $P(\mathcal{J})$, the following statements are equivalent:*

1)
$$\begin{cases} \hat{\mu}(A,B) = \hat{\mu}(A,\emptyset) + \hat{\mu}(\emptyset,B) & \textit{for all } (A,B) \in P(\mathcal{J}), & [C1] \\ \hat{\mu}(A,\emptyset) = -\hat{\mu}(\emptyset,A) & \textit{for all } A \subseteq \mathcal{J}, & [C2] \\ & \textit{and one between} & \\ \hat{\mu}(A\cup B,\emptyset) = \hat{\mu}(A,\emptyset) + \hat{\mu}(B,\emptyset) & \textit{for all } A,B \subseteq \mathcal{J} : A\cap B = \emptyset. & [C3] \\ \hat{\mu}(\emptyset,A\cup B) = \hat{\mu}(\emptyset,A) + \hat{\mu}(\emptyset,B) & \textit{for all } A,B \subseteq \mathcal{J} : A\cap B = \emptyset. & [C3'] \end{cases}$$

2) there exists $w_j \geq 0$, such that $\sum_{j\in\mathcal{J}} w_j = 1$ and $\hat{\mu}(A,B) = \sum_{j\in A} w_j - \sum_{j\in B} w_j$,

3) Given $w_j \geq 0$ such that $\sum_{j\in\mathcal{J}} w_j = 1$, $\sum_{j\in\mathcal{J}} w_j x_j = Ch^B(x,\hat{\mu})$, for all $x \in \mathbb{R}^n$.

Proof. We shall prove that 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 2) $\Rightarrow$ 1) and therefore the three statements are equivalent.

1) $\Rightarrow$ 2)

By [C1] and [C3], it is obvious that $\hat{\mu}(A,B) = \sum_{j\in A} \hat{\mu}(\{j\},\emptyset) + \sum_{j\in B} \hat{\mu}(\emptyset,\{j\})$. Putting $w_j = \hat{\mu}(\{j\},\emptyset)$ for all $j \in \mathcal{J}$ we have that $w_j = \hat{\mu}(\{j\},\emptyset) \geq \hat{\mu}(\emptyset,\emptyset) = 0$ for monotonicity and normalization conditions on the bicapacity $\hat{\mu}$ while $\sum_{j\in\mathcal{J}} w_j = \sum_{j\in\mathcal{J}} \hat{\mu}(\{j\},\emptyset) = \hat{\mu}(\mathcal{J},\emptyset) = 1$ for the normalization conditions on the bicapacity $\hat{\mu}$. Besides, by [C2] we get

$$\hat{\mu}(A,B) = \sum_{j\in A} \hat{\mu}(\{j\},\emptyset) + \sum_{j\in B} \hat{\mu}(\emptyset,\{j\}) = \sum_{j\in A} \hat{\mu}(\{j\},\emptyset) - \sum_{j\in B} \hat{\mu}(\{j\},\emptyset) = \sum_{j\in A} w_j - \sum_{j\in B} w_j$$

being the thesis.

2) $\Rightarrow$ 3)

For any $x \in \mathbb{R}^n$, after reordering the absolute values of their components in a non-decreasing way $\left(|x_{(1)}| \leq |x_{(2)}| \leq \ldots \leq |x_{(n)}|\right)$, we get its bipolar Choquet integral with respect to the bicapacity $\hat{\mu}$

$$Ch^B(x,\hat{\mu}) = \sum_{j \in \mathcal{J}^>} |x_{(j)}| \Big[\hat{\mu}\left(C_{(j)}, D_{(j)}\right) - \hat{\mu}\left(C_{(j+1)}, D_{(j+1)}\right)\Big]$$

where $C_{(j)} = \{i \in \mathcal{J}^> \ : \ x_i \geq |x_{(j)}|\}$, $D_{(j)} = \{i \in \mathcal{J}^> \ : \ -x_i \geq |x_{(j)}|\}$, and $C_{(n+1)} = D_{(n+1)} = \emptyset$.

Therefore,

$$Ch^B(x,\hat{\mu}) = \sum_{j \in \mathcal{J}^>} |x_{(j)}| \Big[\sum_{i:\ x_i \geq |x_{(j)}|} w_i - \sum_{i:\ -x_i \geq |x_{(j)}|} w_i - \sum_{i:\ x_i \geq |x_{(j+1)}|} w_i + \sum_{i:\ -x_i \geq |x_{(j+1)}|} w_i \Big] =$$

$$= \sum_{j:\ |x_{(j)}| < |x_{(j+1)}|} |x_{(j)}| \Big[\sum_{i:\ x_i \geq |x_{(j)}|} w_i - \sum_{i:\ -x_i \geq |x_{(j)}|} w_i - \sum_{i:\ x_i \geq |x_{(j+1)}|} w_i + \sum_{i:-\ x_i \geq |x_{(j+1)}|} w_i \Big] =$$

$$= \sum_{j: |x_{(j)}| < |x_{(j+1)}|} |x_{(j)}| \Big[\sum_{i:\ x_i = |x_{(j)}|} w_i - \sum_{i:\ -x_i = |x_{(j)}|} w_i \Big] = \sum_{j \in \mathcal{J}} x_j w_j,$$

being the thesis.

3) $\Rightarrow$ 2)

Let us consider $A, B \subseteq \mathcal{J}$ such that $A \cap B = \emptyset$ and the vector $\overline{x} = (1_A, -1_B, 0_{(A \cup B)^C})$ that is the vector having $\overline{x}_j = 1$ if $j \in A$, $\overline{x}_j = -1$ if $j \in B$ and $\overline{x}_j = 0$ otherwise. It is easy observing that $Ch^B(\overline{x},\hat{\mu}) = \hat{\mu}(A,B)$ and $\sum\limits_{j \in \mathcal{J}} w_j \overline{x}_j = \sum\limits_{j \in A} w_j - \sum\limits_{j \in B} w_j$. Therefore, by holding 3), we get $\hat{\mu}(A,B) = \sum\limits_{j \in A} w_j - \sum\limits_{j \in B} w_j$ being the thesis.

2) $\Rightarrow$ 1)

It is straightforward proving that 2) implies 1).

$\square$

Note 3.1. *Let us observe that point 1) of the previous Proposition translates exactly properties expected in the classical case. Indeed:*

- *The reasons in favor and against the preference of an alternative a over an alternative b are taken into account simultaneously without any antagonistic effect (condition [C1]),*
- *The importance of a coalition of criteria is always the same independently if it is in favor or against the preference of a over b (condition [C2]),*

- *There is no synergy or redundancy neither between the criteria being in favor of the preference of a over b nor between the criteria being against this preference (condition [C3] or [C3']. In fact, let us observe that in presence of condition [C2], conditions [C3] and [C3'] are equivalent).*

By observing that

$$\pi(a,b) - \pi(b,a) = \sum_{j\in\mathcal{J}} w_j P_j(a,b) - \sum_{j\in\mathcal{J}} w_j P_j(b,a) = \sum_{j\in\mathcal{J}} w_j \left[P_j(a,b) - P_j(b,a)\right] = \sum_{j\in\mathcal{J}} w_j P_j^B(a,b)$$

and that in the classical case conditions [C1], [C2] and [C3] (or [C3']) are satisfied (see Note 3.1), we get by Proposition 3.1 that $\pi(a,b) - \pi(b,a) = Ch^B(P^B(a,b),\hat{\mu})$.

While $Ch^B(P^B(a,b),\hat{\mu})$ is therefore equivalent to $\pi(a,b)-\pi(b,a)$ in the classical PROMETHEE methods, $Ch^{B+}(P^B(a,b),\hat{\mu})$ and $Ch^{B-}(P^B(a,b),\hat{\mu})$ give the measure of the reasons in favor of the preference of a over b and the measure of the reasons against the preference of a over b, respectively.
By equations (7), (8) and (9) we easily get

$$Ch^B(P^B(a,b),\hat{\mu}) = Ch^{B+}(P^B(a,b),\hat{\mu}) - Ch^{B-}(P^B(a,b),\hat{\mu}) \quad \text{for all } a,b \in A; \tag{10}$$

moreover, for each alternative $a \in A$, we can define the bipolar positive flow, the bipolar negative flow and the bipolar net flow as follows:

$$\phi^{B+}(a) = \frac{1}{m-1} \sum_{b\in A\setminus\{a\}} Ch^{B+}(P^B(a,b),\hat{\mu}) \tag{11}$$

$$\phi^{B-}(a) = \frac{1}{m-1} \sum_{b\in A\setminus\{a\}} Ch^{B-}(P^B(a,b),\hat{\mu}) \tag{12}$$

$$\phi^{B}(a) = \frac{1}{m-1} \sum_{b\in A\setminus\{a\}} Ch^{B}(P^B(a,b),\hat{\mu}) \tag{13}$$

By equation (10), it follows that $\phi^B(a) = \phi^{B+}(a) - \phi^{B-}(a)$ for each $a \in A$.

Analogously to the classical PROMETHEE I and II methods, using the bipolar positive, negative and net flows we propose the bipolar PROMETHEE I and the bipolar PROMETHEE II methods. Given a pair of alternatives $(a,b) \in A \times A$, the bipolar PROMETHEE I method defines a partial order on the set of alternatives A considering a preference ($\mathcal{P}^I_B$), an indifference ($\mathcal{I}^I_B$) and an incomparability ($\mathcal{R}^I_B$) relation defined as follows:

$$
\begin{cases}
a\mathcal{P}_B^I b \quad \text{iff} & \begin{cases} \Phi^{B+}(a) \geq \Phi^{B+}(b), \\ \Phi^{B-}(a) \leq \Phi^{B-}(b), \\ \Phi^{B+}(a) - \Phi^{B-}(a) > \Phi^{B+}(b) - \Phi^{B-}(b) \end{cases} \\
a\mathcal{I}_B^I b \quad \text{iff} & \begin{cases} \Phi^{B+}(a) = \Phi^{B+}(b), \\ \Phi^{B-}(a) = \Phi^{B-}(b) \end{cases} \\
a\mathcal{R}_B^I b \quad \text{otherwise} &
\end{cases}
$$

Given a pair of alternatives $(a,b) \in A \times A$, the bipolar PROMETHEE II method provides, instead, a complete order on A, defining a preference ($\mathcal{P}_B^{II}$) and an indifference ($\mathcal{I}_B^{II}$) relation as follows: $aP_B^{II}b$ iff $\Phi^B(a) > \Phi^B(b)$, while $aI_B^{II}b$ iff $\Phi^B(a) = \Phi^B(b)$.

3.1 Symmetry conditions

Because $Ch^B(P^B(a,b),\hat{\mu})$ is equivalent to $\pi(a,b) - \pi(b,a) = P^C(a,b)$ in the classical PROMETHEE method, it is reasonable expecting that, for all $a,b \in A$, $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$. The following Proposition gives conditions to satisfy such a requirement:

Proposition 3.2. *$Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$ for all possible a,b, iff*

$$\hat{\mu}(C,D) = -\hat{\mu}(D,C) \text{ for each } (C,D) \in P(\mathcal{J}) \qquad [C4]$$

Proof. See Appendix. □

Observe that condition [C4] considered in the Proposition 3.2 is a generalization of the condition [C2] of Proposition 3.1; moreover, Proposition 3.1 still works if condition [C2] is replaced by condition [C4]. Analogously, because $Ch^{B+}(P^B(a,b),\hat{\mu})$ represents how much a is preferred to b and $Ch^{B-}(P^B(b,a),\hat{\mu})$ represents how much b is preferred to a, it is reasonable to expect that $Ch^{B+}(P^B(a,b),\hat{\mu}) = Ch^{B-}(P^B(b,a),\hat{\mu})$. Sufficient and necessary conditions to get this equality are given by the following Proposition.

Proposition 3.3. *$Ch^{B+}(P^B(a,b),\hat{\mu}) = Ch^{B-}(P^B(b,a),\hat{\mu})$ for all possible a,b, iff $\mu^+(C,D) = \mu^-(D,C)$ for each $(C,D) \in P(\mathcal{J})$.*

Proof. Analogous to Proposition 3.2. □

Reminding equation (10), the Corollary follows.

Corollary 3.1. *$Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$ for all possible a,b, if $\mu^+(C,D) = \mu^-(D,C)$ for each $(C,D) \in P(\mathcal{J})$.*

Proof. See Appendix. □

3.2 The 2-additive decomposable bipolar PROMETHEE methods

As seen in the previous section, the use of the bipolar Choquet integral is based on a bicapacity which assigns numerical values to each element $P(\mathcal{J})$. Let us remark that the number of elements of $P(\mathcal{J})$ is 3^n. This means that the definition of a bicapacity requires a rather huge and unpractical number of parameters. Moreover, the interpretation of these parameters is not always simple for the DM. Therefore, the use of the bipolar Choquet integral in real-world decision-making problems requires some methodology to assist the DM in assessing the preference parameters (bicapacities). Several studies dealing with the determination of the preference parameters representing the relative importance of criteria were proposed in MCDA (see e.g. Roy and Mousseau 1996). The question of the determination of preference parameters representing importance of criteria in presence of interaction between criteria was also studied in the context of MAUT considering the Choquet integral as value function (Angilella et al., 2004, 2010; Marichal and Roubens, 2000).

In the following we consider only the 2-additive bicapacities (Grabisch and Labreuche, 2005a; Fujimoto, 2004), being a particular class of bicapacities.

3.3 Defining a manageable and meaningful bicapacity measure

According to Greco and Figueira (2003), we give the following decomposition of the functions μ^+ and μ^- previously defined:

Definition 3.1.

- $\mu^+(C,D) = \sum\limits_{j\in C} a^+(\{j\},\emptyset) + \sum\limits_{\{j,k\}\subseteq C} a^+(\{j,k\},\emptyset) + \sum\limits_{j\in C,\ k\in D} a^+(\{j\},\{k\})$
- $\mu^-(C,D) = \sum\limits_{j\in D} a^-(\emptyset,\{j\}) + \sum\limits_{\{j,k\}\subseteq D} a^-(\emptyset,\{j,k\}) + \sum\limits_{j\in C,\ k\in D} a^-(\{j\},\{k\})$

The interpretation of each $a^\pm(\cdot)$ is the following:

- $a^+(\{j\},\emptyset)$, represents the power of criterion g_j by itself; this value is always non negative;
- $a^+(\{j,k\},\emptyset)$, represents the interaction between g_j and g_k, when they are in favor of the preference of a over b; when its value is zero there is no interaction; on the contrary, when the value is positive there is a synergy effect when putting together g_j and g_k; a negative value means that the two criteria are redundant;
- $a^+(\{j\},\{k\})$, represents the power of criterion g_k against criterion g_j, when criterion g_j is in favor of a over b and g_k is against to the preference of a over b; this leads always to a reduction or no effect on the value of μ^+ since this value is always non-positive.

An analogous interpretation can be applied to the values $a^-(\emptyset,\{j\})$, $a^-(\emptyset,\{j,k\})$, and $a^-(\{j\},\{k\})$.

For the sake of simplicity, in what follows we will use a_j^+, a_{jk}^+, $a_{j|k}^+$ instead of $a^+(\{j\},\emptyset)$, $a^+(\{j,k\},\emptyset)$, $a^+(\{j\},\{k\})$, and a_j^-, a_{jk}^-, $a_{j|k}^-$ instead of $a^-(\emptyset,\{j\})$, $a^-(\emptyset,\{j,k\})$ and $a^-(\{j\},\{k\})$.

In this way, the bicapacity $\hat{\mu}$, decomposed using μ^+ and μ^- of Definition 3.1, has the following expression:

$$\begin{aligned}\hat{\mu}(C,D) &= \mu^+(C,D)-\mu^-(C,D)= \\ &= \sum_{j\in C} a_j^+ - \sum_{j\in D} a_j^- + \sum_{\{j,k\}\subseteq C} a_{jk}^+ - \sum_{\{j,k\}\subseteq D} a_{jk}^- + \sum_{j\in C,\, k\in D} a_{j|k}^+ - \sum_{j\in C,\, k\in D} a_{j|k}^- \end{aligned} \tag{14}$$

We call such a bicapacity $\hat{\mu}$, a *2-additive decomposable bicapacity* . An analogous decomposition has been proposed directly for $\hat{\mu}$ without considering μ^+ and μ^- in Fujimoto and Murofushi 2005 and Fujimoto et al. 2007, where a function $b: P(\mathcal{J}) \rightarrow \mathbb{R}$ has been considered such that

$$\hat{\mu}(A,B)=\sum_{\substack{C\subseteq A\\ D\subseteq B}} b(C,D), \quad \text{for all} \quad (A,B)\in P(\mathcal{J}) \tag{15}$$

(14) corresponds to (15) in case $b(C,D)=0$ if $|C\cup D|>2$ and putting

- $a_j^+ = b(\{j\},\emptyset)$, and $a_j^- = -b(\emptyset,\{j\})$,
- $a_{jk}^+ = b(\{j,k\},\emptyset)$, and $a_{jk}^- = -b(\emptyset,\{j,k\})$,
- $a_{j|k}^+ - a_{j|k}^- = b(\{j\},\{k\})$.

Notice that, decomposition (14) is richer than decomposition (15) because it permits to distinguish between $a_{j|k}^+$ and $a_{j|k}^-$ while in (15) they are compensated in $b(\{j\},\{k\})$. The distinction between $a_{j|k}^+$ and $a_{j|k}^-$ is important in MCDA as shown in Figueira et al. (2009a) with respect to ELECTRE methods. Notice that another decomposition of the bicapacity has been proposed in Grabisch and Labreuche (2005a). A comparison between the decomposition (15) and the decomposition in Grabisch and Labreuche (2005a) is in Fujimoto et al. (2007).

Considering the decompositions of μ^+ and μ^- given in Definition 3.1, the monotonicity conditions (5), (6) and the boundary conditions (3), (4) have to be expressed in function of the parameters a_j^+, a_{jk}^+, $a_{j|k}^+$, a_j^-, a_{jk}^-, and $a_{j|k}^-$ as follows:

Monotonicity conditions

1) $\mu^+(C,D) \leq \mu^+(C \cup \{j\}, D), \quad \forall\, j \in \mathcal{J},\ \forall (C \cup \{j\}, D) \in P(\mathcal{J})$

$$\Leftrightarrow a_j^+ + \sum_{k \in C} a_{jk}^+ + \sum_{k \in D} a_{j|k}^+ \geq 0, \ \ \forall\, j \in \mathcal{J},\ \forall (C \cup \{j\}, D) \in P(\mathcal{J})$$

2) $\mu^+(C,D) \geq \mu^+(C, D \cup \{j\}), \ \forall\, j \in \mathcal{J},\ \forall (C, D \cup \{j\}) \in P(\mathcal{J})$

$$\Leftrightarrow \sum_{k \in C} a_{k|j}^+ \leq 0, \ \ \forall\, j \in \mathcal{J},\ \forall (C, D \cup \{j\}) \in P(\mathcal{J})$$

being already satisfied because $a_{k|j}^+ \leq 0$, $\forall k, j \in \mathcal{J}, k \neq j$.

3) $\mu^-(C,D) \leq \mu^-(C, D \cup \{j\}), \quad \forall\, j \in \mathcal{J},\ \forall (C, D \cup \{j\}) \in P(\mathcal{J})$

$$\Leftrightarrow a_j^- + \sum_{k \in D} a_{jk}^- + \sum_{k \in C} a_{k|j}^- \geq 0, \ \ \forall\, j \in \mathcal{J},\ \forall (C, D \cup \{j\}) \in P(\mathcal{J})$$

4) $\mu^-(C,D) \geq \mu^-(C \cup \{j\}, D), \ \forall\, j \in \mathcal{J},\ \forall (C \cup \{j\}, D) \in P(\mathcal{J})$

$$\Leftrightarrow \sum_{k \in D} a_{j|k}^- \leq 0, \ \ \forall\, j \in \mathcal{J},\ \forall (C \cup \{j\}, D) \in P(\mathcal{J})$$

being already satisfied because $a_{j|k}^- \leq 0$, $\forall j, k \in \mathcal{J}, j \neq k$.

Conditions 1), 2), 3) and 4) ensure the monotonicity of the bi-capacity $\hat{\mu}$, on $\mathcal{J}$, obtained as the difference of μ^+ and μ^-, that is,

$$\forall \ \ (C,D),\ (E,F) \ \in \ P(\mathcal{J}) \ \ \text{such that} \ \ C \supseteq E,\ D \subseteq F,\ \ \hat{\mu}(C,D) \geq \hat{\mu}(E,F).$$

Boundary conditions

1. $\mu^+(\mathcal{J}, \emptyset) = 1$, i.e., $\displaystyle\sum_{j \in \mathcal{J}} a_j^+ + \sum_{\{j,k\} \subseteq \mathcal{J}} a_{jk}^+ = 1$
2. $\mu^-(\emptyset, \mathcal{J}) = 1$, i.e., $\displaystyle\sum_{j \in \mathcal{J}} a_j^- + \sum_{\{j,k\} \subseteq \mathcal{J}} a_{jk}^- = 1$

3.4 The 2-additive bipolar Choquet integral

The following Proposition gives an expression of $Ch^{B+}(x, \hat{\mu})$ and $Ch^{B-}(x, \hat{\mu})$ considering a 2-additive decomposable bicapacity $\hat{\mu}$.

Proposition 3.4. *Given a 2-additive decomposable bicapacity $\hat{\mu}$, then for all $x \in \mathbb{R}^n$*

1. $Ch^{B+}(x,\hat{\mu}) = \sum\limits_{j\in\mathcal{J}, x_j>0} a_j^+ x_j + \sum\limits_{\substack{j,k\in\mathcal{J}, j\neq k\\ x_j,x_k>0}} a_{jk}^+ \min\{x_j,x_k\} + \sum\limits_{\substack{j,k\in\mathcal{J}, j\neq k\\ x_j>0,\ x_k<0}} a_{j|k}^+ \min\{x_j,-x_k\}$

2. $Ch^{B-}(x,\hat{\mu}) = -\sum\limits_{j\in\mathcal{J}, x_j<0} a_j^- x_j - \sum\limits_{\substack{j,k\in\mathcal{J}, j\neq k\\ x_j,x_k<0}} a_{jk}^- \max\{x_j,x_k\} - \sum\limits_{\substack{j,k\in\mathcal{J}, j\neq k\\ x_j>0,\ x_k<0}} a_{j|k}^- \max\{-x_j,x_k\}$

Proof. See Appendix.

□

A result analogous to that one given by Proposition 3.4 was already provided in Fujimoto et al. (2007) with respect to the Choquet integral expressed by means of a bicapacity. Our result is slightly different because it refers to the case in which the bicapacity $\hat{\mu}$ can be decomposed in the difference between μ^+ and μ^-.

In the following, we provide the symmetry conditions of Propositions 3.2 and 3.3 in terms of the parameters a_j^+, a_j^-, a_{jk}^+, a_{jk}^-, $a_{j|k}^+$ and $a_{j|k}^-$.

Proposition 3.5. *Given a 2-additive decomposable bicapacity $\hat{\mu}$, then $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ for each $(C,D) \in P(\mathcal{J})$ iff*

1. *for each $j \in \mathcal{J}$, $a_j^+ = a_j^-$,*

2. *for each $\{j,k\} \subseteq \mathcal{J}$, $a_{jk}^+ = a_{jk}^-$,*

3. *for each $j,k \in \mathcal{J}$, $j \neq k$, $a_{j|k}^+ - a_{j|k}^- = a_{k|j}^- - a_{k|j}^+$.*

Proof. See Appendix. □

Corollary 3.2. *Given a 2-additive decomposable bicapacity $\hat{\mu}$, $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$ for all $a,b \in A$ iff*

1. *for each $j \in \mathcal{J}$, $a_j^+ = a_j^-$,*

2. *for each $\{j,k\} \subseteq \mathcal{J}$, $a_{jk}^+ = a_{jk}^-$,*

3. *for each $j,k \in \mathcal{J}$, $j \neq k$, $a_{j|k}^+ - a_{j|k}^- = a_{k|j}^- - a_{k|j}^+$.*

Proof. It follows by Propositions 3.5 and 3.2. □

Proposition 3.6. *Given a 2-additive decomposable bicapacity $\hat{\mu}$, then $\mu^+(C,D) = \mu^-(D,C)$ for each $(C,D) \in P(\mathcal{J})$ iff*

1. *for each* $j \in \mathcal{J}$, $a_j^+ = a_j^-$,

2. *for each* $\{j,k\} \subseteq \mathcal{J}$, $a_{jk}^+ = a_{jk}^-$,

3. *for each* $j,k \in \mathcal{J}$, $j \neq k$, $a_{j|k}^+ = a_{k|j}^-$.

Proof. Analogous to Proposition 3.5. □

Corollary 3.3. *Given a 2-additive decomposable bicapacity* $\hat{\mu}$, $Ch^{B+}(P^B(a,b),\hat{\mu}) = Ch^{B-}(P^B(b,a),\hat{\mu})$ *for all* $a,b \in A$ *iff*

1. *for each* $j \in \mathcal{J}$, $a_j^+ = a_j^-$,

2. *for each* $\{j,k\} \subseteq \mathcal{J}$, $a_{jk}^+ = a_{jk}^-$,

3. *for each* $j,k \in \mathcal{J}$, $j \neq k$, $a_{j|k}^+ = a_{k|j}^-$.

Proof. It follows by Propositions 3.6 and 3.3. □

Because the first two conditions of Proposition 3.5 are the same of the first two conditions of Proposition 3.6 but, the third condition of Proposition 3.6 implies the third one of Proposition 3.5, in order to get both $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$ and $Ch^{B+}(P^B(a,b),\hat{\mu}) = Ch^{B-}(P^B(b,a),\hat{\mu})$ for all $a,b \in A$, we impose that should be fulfilled the conditions in Proposition 3.6.

4 Eliciting the preference information

On the basis of the considered 2-additive decomposable bicapacity $\hat{\mu}$, and holding the symmetry condition in Corollary 3.3, we propose the following methodology which simplifies the assessment of the preference information.

We consider the following information provided by the DM and their representation in terms of linear constraints:

1. *Comparing pairs of actions locally or globally.* The constraints represent some pairwise comparisons on a set of training actions. Given two actions a and b, the DM may prefer a to b, b to a or be indifferent to both:

   (a) the linear constraint associated with $a\mathcal{P}b$ (a is locally preferred to b) is:

$$Ch^B(P^B(a,b),\hat{\mu}) > 0;$$

(b) the linear constraints associated with $a\mathcal{P}_B^I b$ (a is preferred to b with respect to the bipolar PROMETHEE I method) are:

$$\left.\begin{array}{l}\Phi^{B+}(a) \geq \Phi^{B+}(b),\\ \Phi^{B-}(a) \leq \Phi^{B-}(b),\\ \Phi^{B+}(a) - \Phi^{B-}(a) > \Phi^{B+}(b) - \Phi^{B-}(b),\end{array}\right\}$$

(c) the linear constraint associated with $a\mathcal{P}_B^{II} b$ (a is preferred to b with respect to the bipolar PROMETHEE II method) is:

$$\Phi^B(a) > \Phi^B(b)$$

(d) the linear constraint associated with $a\mathcal{I}b$ (a is locally indifferent to b) is:

$$Ch^B(P^B(a,b),\hat{\mu}) = 0$$

(e) the linear constraints associated with $a\mathcal{I}_B^I b$ (a is indifferent to b with respect to the bipolar PROMETHEE I method) are:

$$\left.\begin{array}{l}\Phi^{B+}(a) = \Phi^{B+}(b),\\ \Phi^{B-}(a) = \Phi^{B-}(b),\end{array}\right\}$$

(f) the linear constraint associated with $a\mathcal{I}_B^{II} b$ (a is indifferent to b with respect to the bipolar PROMETHEE II method) is:

$$\Phi^B(a) = \Phi^B(b)$$

2. *Comparison of the intensity of preferences between pairs of actions.* In some cases, the DM is able to provide information on intensity of preference. For example, she could state that a is strongly preferred to b while c is weakly preferred to d and this situation could be summarized saying that a is preferred to b more than c is preferred to d (see, for example, Debreu 1979, Dyer and Sarin 1979, Bana e Costa and Vansnick 1994, Figueira et al. 2009b).
Given four actions a, b, c and d:

(a) the linear constraints associated with $(a,b)\mathcal{P}(c,d)$ (the local preference of a over b is larger than the local preference of c over d) is:

$$Ch^B(P^B(a,b),\hat{\mu}) > Ch^B(P^B(c,d),\hat{\mu})$$

(b) the linear constraints associated with $(a,b)\mathcal{I}(c,d)$ (the local preference of a over b is the same of the local preference of c over d) is:

$$Ch^B(P^B(a,b),\hat{\mu}) = Ch^B(P^B(c,d),\hat{\mu})$$

3. *Importance of criteria.* A partial ranking over the set of criteria $\mathcal{J}$ may be provided by the DM:
   (a) criterion g_j is more important than criterion g_k, which leads to the constraint $a_j > a_k$;
   (b) criterion g_j is equally important to criterion g_k, which leads to the constraint $a_j = a_k$.
4. *The sign of interactions.* The DM may be able, in certain cases, to provide the sign of some interactions. For example, if there is a synergy effect when criterion g_j interacts with criterion g_k, the following constraint should be added to the model: $a_{jk} > 0$.
5. *Interaction between pairs of criteria.* The DM can provide some information about interaction between criteria:
   a) if the DM feels that interaction between g_j and g_k is greater than the interaction between g_p and g_q, the constraint should be defined as follows: $|a_{jk}| > |a_{pq}|$ where in particular:
      - if both couples of criteria are synergic then: $a_{jk} > a_{pq}$,
      - if both couples of criteria are redundant then: $a_{jk} < a_{pq}$,
      - if (j,k) is a couple of synergic criteria and (p,q) is a couple of redundant criteria, then: $a_{jk} > -a_{pq}$,
      - if (j,k) is a couple of redundant criteria and (p,q) is a couple of synergic criteria, then: $-a_{jk} > a_{pq}$.

   b) if the DM feels that the strength of the interaction between g_j and g_k is the same of the strength of the interaction between g_p and g_q, the constraint will be the following: $|a_{jk}| = |a_{pq}|$ and in particular:
      - if both couples of criteria are synergic or redundant then: $a_{jk} = a_{pq}$,
      - if one couple of criteria is synergic and the other is redundant then: $a_{jk} = -a_{pq}$,
6. *The power of the opposing criteria.* Concerning the power of the opposing criteria several situations may occur. For example:
   a) when the opposing power of g_k is larger than the opposing power of g_h, with respect to g_j, which expresses a positive preference, we can define the following constraint: $a^+_{j|k} < a^+_{j|h}$ (because $a^+_{j|h} \leq 0$ and $a^-_{j|h} \leq 0$ for all j,k with $j \neq k$);

b) if the opposing power of g_k, expressing negative preferences, is larger with g_j rather than with g_h, the constraint will be $a^+_{j|k} < a^+_{h|k}$.

4.1 A linear programming model

All the constraints presented in the previous section along with the symmetry, boundary and monotonicity conditions can now be put together and form a system of linear constraints. Strict inequalities can be converted into weak inequalities by adding a variable ε. It is well-know that such a system has a feasible solution if and only if when maximizing ε, its value is strictly positive (Marichal and Roubens, 2000). Considering constraints given by Corollary 3.3 for the symmetry condition, the linear programming model can be stated as follows (where $j\mathcal{P}k$ means that criterion g_j is more important than criterion g_k; the remaining relations have a similar interpretation):

Max ε

$$\left.\begin{array}{ll}
Ch^B(P^B(a,b),\hat{\mu}) \geq \varepsilon \ \text{ if } \ a\mathcal{P}b, & Ch^B(P^B(a,b),\hat{\mu}) = 0 \ \text{ if } \ a\mathcal{I}b, \\
\left.\begin{array}{l} \Phi^{B+}(a) \geq \Phi^{B+}(b), \\ \Phi^{B-}(a) \leq \Phi^{B-}(b), \\ \Phi^{B+}(a) - \Phi^{B-}(a) \geq \Phi^{B+}(b) - \Phi^{B-}(b) + \varepsilon \end{array}\right\} \text{ if } \ a\mathcal{P}^I_B b & \left.\begin{array}{l} \Phi^{B+}(a) = \Phi^{B+}(b), \\ \Phi^{B-}(a) = \Phi^{B-}(b) \end{array}\right\} \text{ if } \ a\mathcal{I}^I_B b \\
\Phi^B(a) \geq \Phi^B(b) + \varepsilon \ \text{ if } \ a\mathcal{P}^{II}_B b & \Phi^B(a) = \Phi^B(b) \ \text{ if } \ a\mathcal{I}^{II}_B b \\
Ch^B(P^B(a,b),\hat{\mu}) \geq Ch^B(P^B(c,d),\hat{\mu}) + \varepsilon \ \text{ if } \ (a,b)\mathcal{P}(c,d), & Ch^B(P^B(a,b),\hat{\mu}) = Ch^B(P^B(c,d),\hat{\mu}) \ \text{ if } \ (a,b)\mathcal{I}(c,d), \\
a_j - a_k \geq \varepsilon \ \text{ if } \ j\mathcal{P}k, & a_j = a_k \ \text{ if } \ j\mathcal{I}k, \\
|a_{jk}| - |a_{pq}| \geq \varepsilon \ \text{ if } \ \{j,k\}\mathcal{P}\{p,q\}, \ \text{(see point 5.a) of the previous subsection)} & \\
|a_{jk}| = |a_{pq}| \ \text{ if } \ \{j,k\}\mathcal{I}\{p,q\}, \ \text{(see point 5.b) of the previous subsection)} & \\
a_{jk} \geq \varepsilon \ \text{ if there is synergy between criteria } j \text{ and } k, & \\
a_{jk} \leq -\varepsilon \ \text{ if there is redundancy between criteria } j \text{ and } k, & \\
a_{jk} = 0 \ \text{ if criteria } j \text{ and } k \text{ are not interacting,} & \\
\text{Power of the opposing criteria of the type 6:} & \\
a^+_{j|k} - a^+_{j|p} \geq \varepsilon, & a^-_{j|k} - a^-_{j|p} \geq \varepsilon, \\
a^+_{j|k} - a^+_{p|k} \geq \varepsilon, & a^-_{j|k} - a^-_{p|k} \geq \varepsilon, \\
\text{Symmetry conditions (Proposition 3.3):} & \\
a^+_{j|k} = a^-_{k|j}, \ \ \forall\, j,k \in \mathcal{J}, j \neq k & \\
\text{Boundary and monotonicity conditions:} & \\
\displaystyle\sum_{j \in \mathcal{J}} a_j + \sum_{\{j,k\} \subseteq \mathcal{J}} a_{jk} = 1, & \\
a_j \geq 0 \ \ \forall\, j \in \mathcal{J}, & a^+_{j|k}, \ a^-_{j|k} \ \leq 0 \ \ \forall\, j,k \in \mathcal{J}, \\
\displaystyle a_j + \sum_{k \in C} a_{jk} + \sum_{k \in D} a^+_{j|k} \geq 0, \ \ \forall\, j \in \mathcal{J}, \ \forall (C \cup \{j\}, D) \in P(\mathcal{J}), & \\
\displaystyle a_j + \sum_{k \in D} a_{jk} + \sum_{h \in C} a^-_{h|j} \geq 0, \ \ \forall\, j \in \mathcal{J}, \ \forall (C, D \cup \{j\}) \in P(\mathcal{J}). &
\end{array}\right\} E^{A^R}$$

4.2 Restoring PROMETHEE

The condition which allows to restore the classical PROMETHEE methods is the following:

1. $\forall j, k \in \mathcal{J}, \;\; a_{jk} = a^{+}_{j|k} = a^{-}_{j|k} = 0.$

If Condition 1. is not satisfied and the following condition holds

2. $\forall j, k \in \mathcal{J}, a^{+}_{j|k} = a^{-}_{j|k} = 0,$

then the comprehensive preference of a over b is calculated as the difference between the Choquet integral of the positive preferences and the Choquet integral of the negative preferences, with a common capacity μ on $\mathcal{J}$ for the positive and the negative preferences, i.e. there exists $\mu : 2^{\mathcal{J}} \rightarrow [0,1]$, with $\mu(\emptyset) = 0$, $\mu(\mathcal{J}) = 1$, and $\mu(A) \leq \mu(B)$ for all $A \subseteq B \subseteq \mathcal{J}$, such that

$$Ch^B(P^B(a,b), \hat{\mu}) = \int_0^1 \mu(\{j \in \mathcal{J} : P^B_j(a,b) > t\})dt - \int_0^1 \mu(\{j \in \mathcal{J} : P^B_j(a,b) < -t\})dt.$$

We shall call this type of aggregation of preferences, the symmetric Choquet integral PROMETHEE method. Let us remember that the symmetric Choquet integral is also known as the Šipoš integral (Šipoš, 1979). If neither 1. nor 2. are satisfied, but the following condition holds

3. $\forall j, k \in \mathcal{J}, a^{+}_{j|k} = a^{-}_{k|j},$

then we have the Bipolar PROMETHEE methods.

A discussion about the relationship between the bipolar Choquet integral, the Šipoš integral and other model translating bipolar preferences, as the Cumulative Prospect Theory (CPT) (Tversky and Kahneman, 1992), go outside the aim of this paper but we invite the interested reader to look at Grabisch and Labreuche (2010) where these relationships are deeply treated.

4.3 A constructive learning preference information elicitation process

The previous Conditions 1.-3. suggest a proper way to deal with the linear programming model in order to assess the interactive bipolar criteria coefficients. Indeed, it is very wise trying before to elicit weights concordant with the classical PROMETHEE method. If this is not possible, one can consider a PROMETHEE method which aggregates positive and negative preferences using the Choquet integral that is the symmetric Choquet integral PROMETHEE method. If, by proceeding in this way, we are not able to represent the

DM's preferences, then we can take into account a more sophisticated aggregation procedure by using the bipolar PROMETHEE method. This way to progress from the simplest to the most sophisticated model can be outlined in a four steps procedure as follows:

1. Solve the linear programming problem

$$\begin{aligned}&\text{Max } \varepsilon = \varepsilon_1 \\ &\left.\begin{aligned}&E^{A^R} \\ &a_{jk} = a^+_{j|k} = a^-_{j|k} = 0, \quad \forall j,k \in \mathcal{J}\end{aligned}\right\} E_1\end{aligned} \tag{16}$$

   adding to E^{A^R} the constraint related to the previous Condition 1. If E_1 is feasible and $\varepsilon_1 > 0$, then the obtained preferential parameters are concordant with the classical PROMETHEE method. Otherwise,

2. Solve the linear programming problem

$$\begin{aligned}&\text{Max } \varepsilon = \varepsilon_2 \\ &\left.\begin{aligned}&E^{A^R} \\ &a^+_{j|k} = a^-_{j|k} = 0, \quad \forall j,k \in \mathcal{J}\end{aligned}\right\} E_2\end{aligned} \tag{17}$$

   adding to E^{A^R} the constraint related to the previous Condition 2. If E_2 is feasible and $\varepsilon_2 > 0$, then the information is concordant with the symmetric Choquet integral PROMETHEE method having a unique capacity for the negative and the positive part. Otherwise,

3. Solve the linear programming problem

$$\begin{aligned}&\text{Max } \varepsilon = \varepsilon_3 \\ &E^{A^R}\end{aligned} \tag{18}$$

   If E_3 is feasible and $\varepsilon_3 > 0$, then the information is concordant with the bipolar PROMETHEE method. Otherwise,

4. We can try to help the DM by providing some information about inconsistent judgments, when it is the case, by using a constructive learning procedure analogous to that one proposed in Mousseau et al. (2003). In fact, in the linear programming model some of the constraints cannot be relaxed, that is, the basic properties of the model (symmetry, boundary and monotonicity conditions). The remaining constraints can lead to an infeasible linear system which means that the DM provided inconsistent

information about her/his preferences. The methods proposed in Mousseau et al. (2003) can then be used in this context, providing the DM some useful information about inconsistent judgments.

5 ROR and Bipolar PROMETHEE methods

In the previous sections we dealt with the problem of finding a bicapacity restoring preference information provided by the DM in case multiple criteria evaluations are aggregated by the Bipolar PROMETHEE method. Generally, there could exist more than one model (in our case the model will be a bicapacity, but in other contexts it could be a utility function or an outranking relation) compatible with the preference information provided by the DM on the training set of alternatives. Each compatible model restores the preference information provided by the DM but two different compatible models could compare the other alternatives not provided as examples by the DM in a different way. For this reason, the choice of one of these models among those compatible could be considered arbitrary. To take into account not only one but the whole set of models compatible with the preference information provided by the DM, we consider the ROR (Greco et al., 2010). This approach considers the whole set of models compatible with the preference information provided by the DM building two preference relations: the weak *necessary* preference relation, for which alternative a is necessarily weakly preferred to alternative b (and we write $a \succsim^N b$), if a is at least as good as b for all compatible models, and the weak *possible* preference relation, for which alternative a is possibly weakly preferred to alternative b (and we write $a \succsim^P b$), if a is at least as good as b for at least one compatible model.

Considering the bipolar flows (11)-(13) and the comprehensive Choquet integral in equation (10), given the alternatives $a, b \in A$, we say that a outranks b (or a is at least as good as b):

- locally, if $Ch^B(P^B(a,b), \hat{\mu}) \geq 0$;
- globally and considering the bipolar PROMETHEE I method, if $\Phi^{B+}(a) \geq \Phi^{B+}(b)$, $\Phi^{B-}(a) \leq \Phi^{B-}(b)$;
- globally and considering the bipolar PROMETHEE II method, if $\Phi^B(a) \geq \Phi^B(b)$.

To check if a is necessarily preferred to b, we look if it is possible that a does not outrank b. Locally, this means that it is possible that there exists a bicapacity $\hat{\mu}$ such that $Ch^B(P^B(a,b), \hat{\mu}) < 0$; globally, considering the bipolar PROMETHEE I method this means that $\Phi^{B+}(a) < \Phi^{B+}(b)$ or $\Phi^{B-}(a) > \Phi^{B-}(b)$, while considering the bipolar PROMETHEE II method this means that $\Phi^B(a) < \Phi^B(b)$. Given the following set of constraints,

$$
\left.\begin{array}{l}
E^{A^R} \\[2ex]
\text{if one verifies the truth of global outranking:} \\[2ex]
\quad \text{if exploited in the way of the bipolar PROMETHEE II method, then:} \\[2ex]
\qquad \Phi^B(a) + \varepsilon \leq \Phi^B(b) \\[2ex]
\quad \text{if exploited in the way of the bipolar PROMETHEE I method, then:} \\[2ex]
\qquad \Phi^{B+}(a) + \varepsilon \leq \Phi^{B+}(b) + 2M_1 \;\; \text{and} \;\; \Phi^{B-}(a) + 2M_2 \geq \Phi^{B-}(b) + \varepsilon \\[2ex]
\qquad\quad \text{where } M_i \in \{0,1\}, i = 1,2, \;\text{and}\; \sum_{i=1}^{2} M_i \leq 1 \\[2ex]
\text{if one verifies the truth of local outranking:} \\[2ex]
\qquad Ch^B(P^B(a,b),\hat{\mu}) + \varepsilon \leq 0
\end{array}\right\} E^N(a,b)
$$

we say that a is weakly necessarily preferred to b if $E^N(a,b)$ is infeasible or $\varepsilon^* \leq 0$ where $\varepsilon^* = \max \varepsilon$ s.t. $E^N(a,b)$.

To check if a is possibly preferred to b, we check if it is possible that a outranks b for at least one bicapacity $\hat{\mu}$. Locally, this means that there exists a bicapacity $\hat{\mu}$ such that $Ch^B(P^B(a,b),\hat{\mu}) \geq 0$; globally, considering the bipolar PROMETHEE I method this means that $\Phi^{B+}(a) \geq \Phi^{B+}(b)$ and $\Phi^{B-}(a) \leq \Phi^{B-}(b)$, while considering the bipolar PROMETHEE II method this means that $\Phi^B(a) \geq \Phi^B(b)$. Given the following set of constraints,

$$
\left.\begin{array}{l}
E^{A^R} \\[2ex]
\text{if one verifies the truth of global outranking:} \\[2ex]
\quad \text{if exploited in the way of the bipolar PROMETHEE II method, then:} \\[2ex]
\qquad \Phi^B(a) \geq \Phi^B(b) \\[2ex]
\quad \text{if exploited in the way of the bipolar PROMETHEE I method, then:} \\[2ex]
\qquad \Phi^{B+}(a) \geq \Phi^{B+}(b) \;\; \text{and} \;\; \Phi^{B-}(a) \leq \Phi^{B-}(b) \\[2ex]
\text{if one verifies the truth of local outranking:} \\[2ex]
\qquad Ch^B(P^B(a,b),\hat{\mu}) \geq 0
\end{array}\right\} E^P(a,b)
$$

we say that a is weakly possibly preferred to b if $E^P(a,b)$ is feasible and $\varepsilon^* > 0$ where $\varepsilon^* = \max \varepsilon$ s.t. $E^P(a,b)$.

6 Didactic example

Inspired by an example in literature (Grabisch, 1996), let us consider the problem of evaluating High School students according to their grades in Mathematics, Physics and Literature. In the following we suppose that the Director is the DM, while we will cover the role of analyst helping and supporting the DM in (her)his evaluations.

The Director thinks that scientific subjects (Mathematics and Physics) are more important than Literature. However, when students a and b are compared, if a is better than b both at Mathematics and Physics but a is much worse than b at Literature, then the Director has some doubts about the comprehensive preference of a over b.

Mathematics and Physics are in some sense *redundant* with respect to the comparison of students, since usually students which are good at Mathematics are also good at Physics. As a consequence, if a is better than b at Mathematics, the comprehensive preference of the student a over the student b is stronger if a is better than b at Literature rather than if a is better than b at Physics.

Let us consider the students whose grades (belonging to the range $[0, 20]$) are represented in Table 1 and the following formulation of the preference of a over b with respect to each criterion g_j, for all $j = (M)$ Mathematics, (Ph) Physics, (L) Literature.

Students	Mathematics	Physics	Literature
s_1	16	16	16
s_2	15	13	18
s_3	19	18	14
s_4	18	16	15
s_5	15	16	17
s_6	13	13	19
s_7	17	19	15
s_8	15	17	16

Table 1: Evaluations of the students

$$P_j(a,b) = \begin{cases} 0 & \text{if} \quad g_j(b) \geq g_j(a) \\ (g_j(a) - g_j(b))/4 & \text{if} \quad 0 < g_j(a) - g_j(b) \leq 4 \\ 1 & \text{otherwise} \end{cases}$$

From the values of the partial preferences $P_j(a,b)$, we obtain the bipolar preference function $P_j^B(a,b)$ with respect to each criterion g_j, for $j = M, Ph, L$ using the definition (1). Thus, to each pair of students (s_i, s_j) is associated a vector of three elements:

$P^B(s_i, s_j) = \left[P_M^B(s_i, s_j), P_{Ph}^B(s_i, s_j), P_L^B(s_i, s_j)\right]$; for example, to the pair of students (s_1, s_2) is associated the vector $P^B(s_1, s_2) = [0.25, 0.75, -0.5]$.

Let us suppose that the Dean provides the following information regarding some pairs of students:

- student s_1 is preferred to student s_2 more than student s_3 is preferred to student s_4,
- student s_7 is preferred to student s_8 more than student s_5 is preferred to student s_6.

As explained in section 4, this information is translated by the constraints:

$$Ch^B(P^B(s_1, s_2), \hat{\mu}) > Ch^B(P^B(s_3, s_4), \hat{\mu}), \text{ and } Ch^B(P^B(s_7, s_8), \hat{\mu}) > Ch^B(P^B(s_5, s_6), \hat{\mu})$$

Following the procedure described in section 4.3, at first we check if the classical PROMETHEE method and the symmetric Choquet integral PROMETHEE method are able to restore the preference information provided by the Dean; solving the optimization problems (16) and (17), we get $\varepsilon_1 < 0$ and $\varepsilon_2 < 0$ and therefore neither the classical PROMETHEE method nor the symmetric Choquet integral PROMETHEE method are able to explain the preferences provided by the Dean. Solving the optimization problem (18), we get $\varepsilon_3 > 0$; this means that the information provided by the Dean can be explained by the Bipolar PROMETHEE method.

In order to better understand the problem at hand, we suggested to the Dean to use the ROR applied to the bipolar PROMETHEE method as discussed in the previous section. Using the first piece of preference information, we get the necessary and possible preference relations shown in Table 2 at local level and considering the bipolar PROMETHEE II and PROMETHEE I methods. In Table 2(a), the value 1 in position (i, j) means that s_i is necessarily locally preferred to s_j while the viceversa corresponds to the value 0. Analogous meaning have the values 1 and 0 in in Tables 2(b) and 2(c) respectively.

Table 2: Necessary preference relations after the first piece of preference information

(a) Local

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	0	0	0	1	0	0
s_2	0	0	0	0	0	0	0	0
s_3	1	1	0	1	0	1	0	0
s_4	0	1	0	0	0	0	0	0
s_5	0	1	0	0	0	1	0	0
s_6	0	0	0	0	0	0	0	0
s_7	1	1	0	0	1	1	0	1
s_8	0	1	0	0	1	1	0	0

(b) Bipolar PROMETHEE II

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	0	0	0	0	0	0	0
s_2	0	0	0	0	0	0	0	0
s_3	0	0	0	1	0	0	0	0
s_4	0	0	0	0	0	0	0	0
s_5	0	1	0	0	0	1	0	0
s_6	0	0	0	0	0	0	0	0
s_7	1	1	0	0	1	1	0	1
s_8	0	0	0	0	0	0	0	0

(c) Bipolar PROMETHEE I

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	0	0	0	0	0	0	0
s_2	0	0	0	0	0	0	0	0
s_3	0	0	0	0	0	0	0	0
s_4	0	0	0	0	0	0	0	0
s_5	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	0	0	0
s_7	1	1	0	0	0	0	0	0
s_8	0	0	0	0	0	0	0	0

Looking at Tables 2, we highlight that s_7, s_3 and s_5 are surely the best among the eight students considered. In fact, s_7 is necessarily preferred to five out of the other seven students both locally and considering the bipolar PROMETHEE II method and, at the same time, (s)he is the only student being necessarily preferred to some other student using the bipolar PROMETHEE I method. s_3 is necessarily preferred to

Table 3: Possible preference relations after the first piece of preference information

(a) Local

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	0	1	1	1	0	1
s_2	0	0	0	0	0	1	0	0
s_3	1	1	0	1	1	1	1	1
s_4	1	1	0	0	1	1	1	1
s_5	1	1	1	1	0	1	0	0
s_6	0	1	0	1	0	0	0	0
s_7	1	1	1	1	1	1	0	1
s_8	1	1	1	1	1	1	0	0

(b) Bipolar PROMETHEE II

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	1	1	1	1	0	1
s_2	1	0	1	1	0	1	0	1
s_3	1	1	0	1	1	1	1	1
s_4	1	1	0	0	1	1	1	1
s_5	1	1	1	1	0	1	0	1
s_6	1	1	1	1	0	0	0	1
s_7	1	1	1	1	1	1	0	1
s_8	1	1	1	1	1	1	0	0

(c) Bipolar PROMETHEE I

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	1	1	1	1	0	1
s_2	0	0	0	1	0	1	0	0
s_3	1	1	0	1	1	1	1	1
s_4	1	1	0	0	1	1	1	1
s_5	1	1	1	1	0	1	0	1
s_6	1	1	1	1	0	0	0	0
s_7	1	1	1	1	1	1	0	1
s_8	1	1	1	1	1	1	0	0

four out of the other seven students locally, and (s)he is necessarily preferred to s_4 considering the bipolar PROMETHEE II method. At the same time, (s)he is locally possibly preferred to s_7 (see Table 3). s_5 is necessarily preferred to s_2 and s_6 considering the bipolar PROMETHEE II method. In order to get a more insight on the problem at hand, we suggest to the Dean to provide other information (s)he is sure about. For this reason, the Dean states that, locally, s_2 is preferred to s_6 and s_8 is preferred to s_1.

Table 4: Necessary preference relations after the second piece of preference information

(a) Local

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	0	0	0	1	0	0
s_2	0	0	0	0	0	$\underline{1}$	0	0
s_3	1	1	0	1	$\underline{1}$	1	0	$\underline{1}$
s_4	$\underline{1}$	1	0	0	0	0	0	0
s_5	0	1	0	0	0	1	0	0
s_6	0	0	0	0	0	0	0	0
s_7	1	1	0	$\underline{1}$	1	1	0	1
s_8	$\underline{1}$	1	0	0	1	1	0	0

(b) Bipolar PROMETHEE II

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	0	0	0	0	0	0	0
s_2	0	0	0	0	0	0	0	0
s_3	0	0	0	1	0	0	0	0
s_4	0	0	0	0	0	0	0	0
s_5	0	1	0	0	0	1	0	0
s_6	0	0	0	0	0	0	0	0
s_7	1	1	0	$\underline{1}$	1	1	0	1
s_8	0	0	0	0	0	0	0	0

(c) Bipolar PROMETHEE I

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	0	0	0	0	0	0	0
s_2	0	0	0	0	0	0	0	0
s_3	0	0	0	0	0	0	0	0
s_4	0	0	0	0	0	0	0	0
s_5	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	0	0	0
s_7	1	1	0	$\underline{1}$	0	0	0	$\underline{1}$
s_8	0	0	0	0	0	0	0	0

Table 5: Possible preference relations after the second piece of preference information

(a) Local

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	0	$\underline{0}$	1	1	0	$\underline{0}$
s_2	0	0	0	0	0	1	0	0
s_3	1	1	0	1	1	1	1	1
s_4	1	1	0	0	1	1	$\underline{0}$	1
s_5	1	1	$\underline{0}$	1	0	1	0	0
s_6	0	$\underline{0}$	0	1	0	0	0	0
s_7	1	1	1	1	1	1	0	1
s_8	1	1	$\underline{0}$	1	1	1	0	0

(b) Bipolar PROMETHEE II

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	1	1	1	1	0	1
s_2	1	0	1	1	0	1	0	1
s_3	1	1	0	1	1	1	1	1
s_4	1	1	0	0	1	1	$\underline{0}$	1
s_5	1	1	1	1	0	1	0	1
s_6	1	1	1	1	0	0	0	1
s_7	1	1	1	1	1	1	0	1
s_8	1	1	1	1	1	1	0	0

(c) Bipolar PROMETHEE I

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	1	$\underline{0}$	1	1	1	0	1
s_2	0	0	0	1	0	1	0	0
s_3	1	1	0	1	1	1	1	1
s_4	1	1	0	0	1	1	$\underline{0}$	1
s_5	1	1	1	1	0	1	0	1
s_6	$\underline{0}$	1	1	1	0	0	0	0
s_7	1	1	1	1	1	1	0	1
s_8	1	1	1	1	1	1	0	0

Translating these preference information using the constraints $Ch^B(P^B(2,6),\hat{\mu}) > 0$ and $Ch^B(P^B(8,1),\hat{\mu}) > 0$, and computing again the necessary and possible preference relations locally and considering both the bipolar PROMETHEE methods, we get the results shown in Tables 4 and 5. In these Tables, underlined cells correspond to new information we have got using the second piece of information provided by the Dean. In particular, in Tables 4 the cell in correspondence of the pair of students (s_i, s_j) is underlined if s_i was not necessarily preferred to s_j after the first iteration, but s_i is necessarily preferred to s_j after the second iteration; in Tables 5, the cell in correspondence of the pair of students (s_i, s_j) is underlined if s_i was possibly preferred to s_j after the first iteration but s_i is not possibly preferred to s_j after the second iteration anymore. Looking at Tables 4 and 5, the Dean is addressed to consider s_7 as the best student. In fact, even if s_7 and s_3 are locally necessarily preferred to all other six considered students, s_7 is still the only one being necessarily preferred to someone else considering the bipolar PROMETHEE I method. Besides, looking at Table 5, we get that s_3 is the only student being possibly preferred to s_7 locally and with respect to the bipolar PROMETHEE I and PROMETHEE II methods but, at the same time, everyone except s_4, is possibly preferred to s_3 considering the bipolar PROMETHEE I method while four students (s_5, s_6, s_7 and s_8) are possibly preferred to s_3 with respect to the bipolar PROMETHEE I method.

7 Conclusions

In this paper we proposed a generalization of the classical PROMETHEE methods. A basic assumption of PROMETHEE methods is the absence of interaction (synergy, redundancy and antagonism) between criteria. We developed a methodology permitting to take into account interaction between criteria (synergy, redundancy and antagonism effects) within PROMETHEE methods by using the bipolar Choquet integral. In this way we obtained a new method called the Bipolar PROMETHEE method.

The DM can give directly the preferential parameters of the method; however, due to their great number, it is advisable using some indirect procedure to elicit the preferential parameters from some preference information provided by the DM.

Since, in general, there is more than one set of parameters compatible with these preference information, we proposed to use the Robust Ordinal Regression (ROR) to consider the whole family of compatible sets of preferential parameters. We believe that the proposed methodology can be successfully applied to many real world problems where interacting criteria have to be considered; besides, in a companion paper, we propose to apply the SMAA methodology to the classical and to the bipolar PROMETHEE methods (for a survey on SMAA methods see Tervonen and Figueira 2008).

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Appendix

Proof of Proposition 3.2

Let us prove that if $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ for each $(C,D) \in P(\mathcal{J})$, then $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$.

As noticed, $P_j^B(a,b) = -P_j^B(b,a)$ for all $j \in \mathcal{J}$, and consequently $|P_{(j)}^B(a,b)| = |-P_{(j)}^B(b,a)| = |P_{(j)}^B(b,a)|$ for all $j \in \mathcal{J}$.

By this, it follows that:

(α) $C_{(j)}(a,b) = \{i \in \mathcal{J}^> \; : \; P_i^B(a,b) \geq |P_{(j)}^B(a,b)|\} = \{i \in \mathcal{J}^> \; : \; -P_i^B(b,a) \geq |P_{(j)}^B(b,a)|\} =$
$= D_{(j)}(b,a)$;

(β) $D_{(j)}(a,b) = \{i \in \mathcal{J}^> \; : \; -P_i^B(a,b) \geq |P_{(j)}^B(a,b)|\} = \{i \in \mathcal{J}^> \; : \; P_i^B(b,a) \geq |P_{(j)}^B(b,a)|\} =$
$= C_{(j)}(b,a)$.

By (α) and (β) we have that

(γ)
$$\begin{aligned} &Ch^B(P^B(a,b),\hat{\mu}) = \\ &= \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(a,b)| \Big[\hat{\mu}(C_{(j)}(a,b), D_{(j)}(a,b)) - \hat{\mu}(C_{(j+1)}(a,b), D_{(j+1)}(a,b))\Big] = \\ &= \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(b,a)| \Big[\hat{\mu}(D_{(j)}(b,a), C_{(j)}(b,a)) - \hat{\mu}(D_{(j+1)}(b,a), C_{(j+1)}(b,a))\Big]. \end{aligned}$$

Since $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$, $\forall (C,D) \in P(\mathcal{J})$, by ($\gamma$) we have that,

(δ)
$$\begin{aligned} &Ch^B(P^B(b,a),\hat{\mu}) = \\ &= \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(b,a)| \Big[\hat{\mu}(C_{(j)}(b,a), D_{(j)}(b,a)) - \hat{\mu}(C_{(j+1)}(b,a), D_{(j+1)}(b,a))\Big] = \\ &= \sum_{j\in\mathcal{J}^>} |P_{(j)}^B(b,a)| \Big[-\hat{\mu}(D_{(j)}(b,a), C_{(j)}(b,a)) + \hat{\mu}(D_{(j+1)}(b,a), C_{(j+1)}(b,a))\Big] \\ &= -Ch^B(P^B(a,b),\hat{\mu}). \end{aligned}$$

Let us now prove that if $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$, then $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$. Let us consider the pair (a,b) such that,

$$P_j^B(a,b) = \begin{cases} 1 & \text{if} \;\; j \in C \\ -1 & \text{if} \;\; j \in D \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

In this case we have that $Ch^B(P^B(a,b),\hat{\mu}) = \hat{\mu}(C,D)$ and $Ch^B(P^B(b,a),\hat{\mu}) = \hat{\mu}(D,C)$. Thus if $Ch^B(P^B(a,b),\hat{\mu}) = -Ch^B(P^B(b,a),\hat{\mu})$, by (iv) we obtain that $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ and the proof is concluded.

Proof of Corollary 3.1

This can be seen as a Corollary both of Proposition 3.2 and Proposition 3.3. In fact,

- $\mu^+(C,D) = \mu^-(D,C)$ for each $(C,D) \in P(\mathcal{J})$ implies that $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$ for each $(C,D) \in P(\mathcal{J})$, and by Proposition 3.2, it follows the thesis.

- $\mu^+(C,D) = \mu^-(D,C)$ for each $(C,D) \in P(\mathcal{J})$ implies that $Ch^{B+}(P^B(a,b),\hat{\mu}) = Ch^{B-}(P^B(b,a),\hat{\mu})$ (by Proposition 3.3) and from this it follows obviously the thesis by equation (10).

Proof of Proposition 3.4

We shall prove only part 1. Proof of part 2. can be obtained analogously.

If the bicapacity $\hat{\mu}$ is 2−additive decomposable, then

$$Ch^{B+}(x,\hat{\mu}) = \sum_{j\in\mathcal{J}^>} |x_{(j)}|\Big[\mu^+(C_{(j)},D_{(j)}) - \mu^+(C_{(j+1)},D_{(j+1)})\Big] =$$

$$= \sum_{j\in\mathcal{J}^>} |x_{(j)}|\Big[\Big(\sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j)}|} a_k^+ - \sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j+1)}|} a_k^+\Big) +$$
$$+\Big(\sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j)}|} a_{hk}^+ - \sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, x_k\geq|x_{(j+1)}|} a_{hk}^+\Big) +$$
$$+\Big(\sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, -x_k\geq|x_{(j)}|} a_{h|k}^+ - \sum_{h,k\in\mathcal{J}^>, h\neq k, x_h, -x_k\geq|x_{(j+1)}|} a_{h|k}^+\Big)\Big]$$

Let us remark that,

$$a)\quad \Big(\sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j)}|} a_k^+ - \sum_{k\in\mathcal{J}^>, x_k\geq|x_{(j+1)}|} a_k^+\Big) = \begin{cases} \displaystyle\sum_{k\in\mathcal{J}^>, x_k=|x_{(j)}|} a_k^+ & \text{if} \quad |x_{(j)}| < |x_{(j+1)}| \\ 0 & \text{otherwise} \end{cases}$$

$$b)\quad \Big(\sum_{\substack{h,k\in\mathcal{J}^>, h\neq k,\\ x_h, x_k\geq|x_{(j)}|}} a_{hk}^+ - \sum_{\substack{h,k\in\mathcal{J}^>, h\neq k,\\ x_h, x_k\geq|x_{(j+1)}|}} a_{hk}^+\Big) = \begin{cases} \displaystyle\sum_{\substack{h,k\in\mathcal{J}^>, h\neq k,\\ \min\{x_h, x_k\}=|x_{(j)}|}} a_{hk}^+ & \text{if} \quad |x_{(j)}| < |x_{(j+1)}| \\ 0 & \text{otherwise} \end{cases}$$

$$c)\quad \Big(\sum_{\substack{h,k\in\mathcal{J}^>, h\neq k,\\ x_h, -x_k\geq|x_{(j)}|}} a_{h|k}^+ - \sum_{\substack{h,k\in\mathcal{J}^>, h\neq k,\\ x_h, -x_k\geq|x_{(j+1)}|}} a_{h|k}^+\Big) = \begin{cases} \displaystyle\sum_{\substack{h,k\in\mathcal{J}^>, h\neq k,\\ \min\{x_h, -x_k\}=|x_{(j)}|}} a_{h|k}^+ & \text{if} \quad |x_{(j)}| < |x_{(j+1)}| \\ 0 & \text{otherwise} \end{cases}$$

Considering $a) - c)$ we get that:

$$\chi) \quad = \sum_{\substack{j\in\mathcal{J}^>, \\ |x_{(j)}|<|x_{(j+1)}|}} |x_{(j)}| \Big[\sum_{k\in\mathcal{J}^>, x_k=|x_{(j)}|} a_k^+ + \sum_{\substack{h,k\in\mathcal{J}^>, h\neq k, \\ \min\{x_h,x_k\}=|x_{(j)}|}} a_{hk}^+ + \sum_{\substack{h,k\in\mathcal{J}^>, h\neq k, \\ \min\{x_h,-x_k\}=|x_{(j)}|}} a_{h|k}^+ \Big]$$

and from this it follows the thesis.

Proof of Proposition 3.5

First, let us prove that

(*a*) $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$

implies 1., 2. and 3. For each $j \in \mathcal{J}$,

(*b*) $\hat{\mu}(\{j\},\emptyset) = a_j^+$ and $\hat{\mu}(\emptyset,\{j\}) = -a_j^-$

From (*a*) and (*b*) we have,

$$a_j^+ = \hat{\mu}(\{j\},\emptyset) = -\hat{\mu}(\emptyset,\{j\}) = a_j^-$$

which is 1.

For each $\{j,k\} \subseteq \mathcal{J}$ we have that,

(*c*) $\hat{\mu}(\{j,k\},\emptyset) = a_j^+ + a_k^+ + a_{jk}^+$ and $\hat{\mu}(\emptyset,\{j,k\}) = -a_j^- - a_k^- - a_{jk}^-$

Being $\hat{\mu}(\{j,k\},\emptyset) = -\hat{\mu}(\emptyset,\{j,k\})$, and being $a_j^+ = a_j^-$ and $a_k^+ = a_k^-$ by 1., we have that for each $\{j,k\} \subseteq \mathcal{J}$, $a_{jk}^+ = a_{jk}^-$, i.e. 2.

For all $j,k \in \mathcal{J}$ with $j \neq k$, we have:

$$\hat{\mu}(\{j\},\{k\}) = a_j^+ - a_k^- + a_{j|k}^+ - a_{j|k}^-$$

$$\hat{\mu}(\{k\},\{j\}) = a_k^+ - a_j^- + a_{k|j}^+ - a_{k|j}^-$$

Being $\hat{\mu}(\{j\},\{k\}) = -\hat{\mu}(\{k\},\{j\})$ and having proved that $a_j^+ = a_j^-, \forall j$, we obtain that $a_{j|k}^+ - a_{j|k}^- = -a_{k|j}^+ + a_{k|j}^-$ i.e. 3.

It is straightforward to prove that 1., 2., and 3. imply $\hat{\mu}(C,D) = -\hat{\mu}(D,C)$.