Multiple Criteria Hierarchy Process for the Choquet integral

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Abstract. Interaction between criteria and hierarchical structure of criteria are two important issues in Multiple Criteria Decision Analysis (MCDA). Interaction between criteria is often dealt with fuzzy integrals, especially the Choquet integral. To handle the hierarchy of criteria in MCDA, a methodology called Multiple Criteria Hierarchy Process (MCHP) has been recently proposed. It permits consideration of preference relations with respect to a subset of criteria at any level of the hierarchy. In this paper, we propose to apply MCHP to the Choquet integral. In this way, using the Choquet integral and the MCHP, it is possible to compare two alternatives not only globally, but also partially taking into account a particular subset of criteria and the possible interaction between them.

Keywords: Multiple criteria decision aiding, Choquet integral, Multiple Criteria Hierarchy Process.

1 Introduction

In a multiple criteria decision problem (see [5] for a recent state of the art), an alternative a, belonging to a finite set of m alternatives $A = \{a, b, c, \ldots\}$, is evaluated on the basis of a consistent family of n criteria $G = \{g_1, g_2, \ldots, g_n\}$. In our approach we make the assumption that each criterion $g_i\colon A \to \mathbb{R}$ is an interval scale of measurement. From here on, we will use the terms criterion g_i or criterion i interchangeably ($i = 1, 2, \ldots, n$). Without loss of generality, we assume that all the criteria have to be maximized.

The purpose of Multi-Attribute Utility Theory (MAUT) [10] is to represent the preferences of a Decision Maker (DM) on a set of alternatives A by an overall value function $U\colon \mathbb{R}^n \to \mathbb{R}$ with $U(g_1(a), \ldots, g_n(a)) = U(a)$:

- a is indifferent to b $\Leftrightarrow$ $U(a) = U(b)$,
- a is preferred to b $\Leftrightarrow$ $U(a) > U(b)$.

The principal aggregation model of value function is the multiple attribute additive utility [10]:

$$U(a) = u_1(g_1(a)) + u_2(g_2(a)) + \ldots + u_n(g_n(a)) \quad \text{with} \quad a \in A,$$

where u_i are non-decreasing marginal value functions for $i = 1, 2, \ldots, n$.

As it is well-known in literature, the underlying assumption of the preference independence of the multiple attribute additive utility is unrealistic since it doesn't permit to represent interaction between the criteria under consideration. In a decision problem we, usually, distinguish between positive and negative interaction among criteria, representing synergy and redundancy among criteria respectively. In particular, two criteria are synergic (redundant) when the comprehensive importance of these two criteria is greater (smaller) than the sum of the importance of the two criteria considered separately.

Within Multiple Criteria Decision Analysis (MCDA), the interaction of criteria has been considered in a decision model based upon a non-additive integral, *viz.* the Choquet integral [3] (see [7] for a comprehensive survey on the use of non-additive integrals in MCDA).

One of the main drawbacks of the Choquet integral decision model is the elicitation of its parameters representing the importance and interaction between criteria.

A great majority of methods designed for MCDA, assume that all evaluation criteria are considered at the same level, however, it is often the case that a practical application is imposing a hierarchical structure of criteria. For example, in economic ranking, alternatives may be evaluated on indicators which aggregate evaluations on several sub-indicators, and these sub-indicators may aggregate another set of sub-indicators, etc. In this case, the marginal value functions may refer to all levels of the hierarchy, representing values of particular scores of the alternatives on indicators, sub-indicators, sub-sub-indicators, etc. Considering hierarchical, instead of flat, structure of criteria, permits decomposition of a complex decision problem into smaller problems involving less criteria. To handle the hierarchy of criteria, the Multiple Criteria Hierarchy Process (MCHP) [4] could be applied. The basic idea of MCHP relies on consideration of preference relations at each node of the hierarchy tree of criteria. These preference relations concern both the phase of eliciting preference information, and the phase of analyzing a final recommendation by the DM. For example, in a decision problem related to evaluation of students, one can say not only that student a is comprehensively preferred to student b, i.e. $a \succ b$, but also that a is comprehensively preferred to b because a is preferred to b on subsets of subjects (subcriteria) related to Mathematics and Physics, i.e. $a \succ_{Mathematics} b$ and $a \succ_{Physics} b$, even if b is preferred to a on subjects related to Humanities, i.e. $b \succ_{Humanities} a$. Moreover, one can also say that, for example, a is preferred to b on the subset of subjects related to Mathematics because, considering Analysis and Algebra as subjects (sub-criteria) related to Mathematics, a is preferred to b on Analysis, i.e. $a \succ_{Analysis} b$, and this is enough to compensate the fact that b is preferred to a on Algebra, i.e. $b \succ_{Algebra} a$.

In this paper, we apply the MCHP to the Choquet integral.

The paper is organized as follows. In Section 2, we present the basic concepts relative to interaction between criteria and to the Choquet integral. In Section 3, we describe the MCHP. In Section 4, we put together the MCHP and the Choquet integral. Section 5 contains a didactic example in which we describe the application of the new methodology. Some conclusions and future directions of research are presented in Section 6.

2 The Choquet integral decision model

Let 2^G be the power set of G (i.e. the set of all subsets of G); a fuzzy measure (capacity) on G is defined as a set function $\mu : 2^G \to [0,1]$ satisfying the following properties:

1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions),
2a) $\forall\, T \subseteq R \subseteq G,\ \ \mu(T) \leq \mu(R)$ (monotonicity condition).

A fuzzy measure is said to be additive if $\mu(T \cup R) = \mu(T) + \mu(R)$, for any $T, R \subseteq G$ such that $T \cap R = \emptyset$. An additive fuzzy measure is determined uniquely by $\mu(\{1\}), \mu(\{2\}) \ldots, \mu(\{n\})$. In fact, in this case, $\forall\ T \subseteq G$, $\mu(T) = \sum_{i \in T} \mu(\{i\})$. In the other cases, we have to define a value $\mu(T)$ for every subset T of G, obtaining $2^{|G|}$ coefficients values. Therefore, we have to calculate the values of $2^{|G|} - 2$ coefficients, since we know that $\mu(\emptyset) = 0$ and $\mu(G) = 1$.

The Möbius representation of the fuzzy measure μ (see [13]) is defined by the function $a : 2^G \to \mathbb{R}$ (see [14]) such that:

$$\mu(R) = \sum_{T \subseteq R} a(T).$$

Let us observe that if R is a singleton, *i.e.* $R = \{i\}$ with $i = 1, \cdots, n$ then $\mu(\{i\}) = a(\{i\})$.

If R is a couple (non-ordered pair) of criteria, i.e. $R = \{i, j\}$, then $\mu(\{i, j\}) = a(\{i\}) + a(\{j\}) + a(\{i, j\})$.

In general, the Möbius representation $a(R)$ is obtained by $\mu(R)$ in the following way:

$$a(R) = \sum_{T \subseteq R} (-1)^{|R-T|} \mu(T).$$

In terms of Möbius representation (see [2]), properties **1a)** and **2a)** are, respectively, formulated as:

1b) $a(\emptyset) = 0, \ \ \sum_{T \subseteq G} a(T) = 1,$

2b) $\forall\, i \in R$ and $\ \forall R \subseteq G, \ \ \sum_{T \subseteq R} a(T) \geq 0.$

Let us observe that in MCDA, the importance of any criterion $g_i \in G$ should be evaluated considering all its global effects in the decision problem at hand; these effects can be "decomposed" from both theoretical and operational points of view in effects of g_i as single, and in combination with all other criteria. Therefore, a criterion $i \in G$ is important with respect to a fuzzy measure μ not only when it is considered alone, i.e. for the value $\mu(\{i\})$ in itself, but also when it interacts with other criteria from G, i.e. for every value $\mu(T \cup \{i\})$, $T \subseteq G \setminus \{i\}$.

Given $x \in A$ and μ being a fuzzy measure on G, then the *Choquet integral* [3] is defined by:

$$C_\mu(x) = \sum_{i=1}^{n} \left[\left(g_{(i)}(x)\right) - \left(g_{(i-1)}(x)\right) \right] \mu\left(A_i\right), \tag{1}$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that:

$$g_{(1)}(x) \leq g_{(2)}(x) \leq ... \leq g_{(n)}(x),$$

with $A_i = \{(i), ..., (n)\}$, $i = 1, .., n$, and $g_{(0)} = 0$.

The Choquet integral can be redefined in terms of the Möbius representation [6], without reordering the criteria, as:

$$C_\mu(x) = \sum_{T \subseteq G} a(T) \min_{i \in T} g_i(x). \tag{2}$$

One of the main drawbacks of the Choquet integral is the necessity to elicitate and give an adequate interpretation of $2^{|G|} - 2$ parameters. In order to reduce the number of parameters to be computed and to eliminate a too strict description of the interactions among criteria, which is not realistic in many applications, the concept of fuzzy k-additive measure has been considered [8].

A *fuzzy measure* is called *k-additive* if $a(T) = 0$ with $T \subseteq G$, when $|T| > k$. We observe that a 1-additive measure is the common additive fuzzy measure. In many real decision problems, it suffices to consider 2-additive measures. In this case, positive and negative interactions between couples of criteria are modeled without considering the interaction among triples, quadruplets and generally n-tuples, (with $n > 2$) of criteria. From the point of view of MCDA, the use of 2-additive measures is justified by observing that the information on the importance of the single criteria and the interactions between couples of criteria are noteworthy. Moreover, it could be not easy or not straightforward for the DM to provide information on the interactions among three or more criteria during the decision procedure. From a computational point of view, the interest in the 2-additive measures lies in the fact that any decision model needs to evaluate a number $n + \binom{n}{2}$ of parameters (in terms of Möbius representation, a value $a(\{i\})$ for every criterion i and a value $a(\{i,j\})$ for every couple of distinct criteria $\{i,j\}$.) With respect to a 2-additive fuzzy measure, the inverse transformation to obtain the fuzzy measure $\mu(R)$ from the Möbius representation is defined as:

$$\mu(R) = \sum_{i \in R} a(\{i\}) + \sum_{\{i,j\} \subseteq R} a(\{i,j\}), \ \forall R \subseteq G. \tag{3}$$

With regard to 2-additive measures, properties **1b)** and **2b)** have, respectively, the following formulations:

1c) $a(\emptyset) = 0, \ \sum_{i \in G} a(\{i\}) + \sum_{\{i,j\} \subseteq G} a(\{i,j\}) = 1,$

2c) $a(\{i\}) \geq 0, \ \forall i \in G, a(\{i\}) + \sum_{j \in T} a(\{i,j\}) \geq 0, \forall i \in G \text{ and } \forall\, T \subseteq G \setminus \{i\}.$

In this case, the representation of the Choquet integral of $x \in A$ is given by:

$$C_\mu(x) = \sum_{\{i\} \subseteq G} a(\{i\})(g_i(x)) + \sum_{\{i,j\} \subseteq G} a(\{i,j\}) \min\{g_i(x), g_j(x)\}. \qquad (4)$$

Finally, we recall the definitions of the importance and interaction indices for a couple of criteria.

The Shapley value [15] expressing the importance of criterion $i \in G$, is given by:

$$\varphi(i) = \sum_{T \subseteq G : i \notin T} \frac{(|G-T|-1)!|T|!}{|G|!} [\mu(T \cup \{i\}) - \mu(T)],$$

while the *interaction index* [12] expressing the sign and the magnitude of the sinergy in a couple of criteria $\{i,j\} \subseteq G$, is given by

$$\varphi(\{i,j\}) = \sum_{T \subseteq G : i,j \notin T} \frac{(|G-T|-2)!|T|!}{(|G|-1)!} [\mu(T \cup \{i,j\}) - \mu(T \cup \{i\})) - \mu(T \cup \{j\}) + \mu(T)].$$

In case of 2-additive capacities the Shapley value and the interaction index can be expressed as follows:

$$\varphi(\{i\}) = a(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{a(\{i,j\})}{2}, \ \ i \in G, \qquad (5)$$

$$\varphi(\{i,j\}) = a(\{i,j\}). \qquad (6)$$

3 Multiple Criteria Hierarchy Process (MCHP)

In MCHP, a set $\mathcal{G}$ of hierarchically ordered criteria is considered, i.e. all criteria are not considered at the same level, but they are distributed over l different levels (see Figure 1). At level 1, there are first level criteria called root criteria. Each root criterion has its own hierarchy tree. The leaves of each hierarchy tree are at the last level l and they are called elementary subcriteria. Thus, in graph theory terms, the whole hierarchy is a forest. We will use the following notation:

- l is the number of levels in the hierarchy of criteria,

- $\mathcal{G}$ is the set of all criteria at all considered levels,
- $\mathcal{I}_\mathcal{G}$ is the set of indices of particular criteria representing position of criteria in the hierarchy,
- m is the number of the first level criteria, $G_1, \ldots, G_m$,
- $G_\mathbf{r} \in \mathcal{G}$, with $\mathbf{r} = (i_1, \ldots, i_h) \in \mathcal{I}_\mathcal{G}$, denotes a subcriterion of the first level criterion G_{i_1} at level h; the first level criteria are denoted by G_{i_1}, $i_1 = 1, \ldots, m$,
- $n(\mathbf{r})$ is the number of subcriteria of $G_\mathbf{r}$ in the subsequent level, i.e. the direct subcriteria of $G_\mathbf{r}$ are $G_{(\mathbf{r},1)}, \ldots, G_{(\mathbf{r},n(\mathbf{r}))}$,
- $g_\mathbf{t} : A \to \mathbb{R}$, with $\mathbf{t} = (i_1, \ldots, i_l) \in \mathcal{I}_\mathcal{G}$, denotes an elementary subcriterion of the first level criterion G_{i_1}, i.e. a criterion at level l of the hierarchy tree of G_{i_1},
- EL is the set of indices of all elementary subcriteria:

$$EL = \{\mathbf{t} = (i_1, \ldots, i_l) \in \mathcal{I}_\mathcal{G}\} \quad \text{where} \quad \begin{cases} i_1 = 1, \ldots, m \\ i_2 = 1, \ldots, n(i_1) \\ \cdots\cdots \\ i_l = 1, \ldots, n(i_1, \ldots, i_{l-1}) \end{cases}$$

- $E(G_\mathbf{r})$ is the set of indices of elementary subcriteria descending from $G_\mathbf{r}$, i.e.

$$E(G_\mathbf{r}) = \{(\mathbf{r}, i_{h+1}, \ldots, i_l) \in \mathcal{I}_\mathcal{G}\} \quad \text{where} \quad \begin{cases} i_{h+1} = 1, \ldots, n(\mathbf{r}) \\ \cdots\cdots \\ i_l = 1, \ldots, n(\mathbf{r}, i_{h+1}, \ldots, i_{l-1}) \end{cases}$$

  thus, $E(G_\mathbf{r}) \subseteq EL$,
- when $\mathbf{r} = 0$, then by $G_\mathbf{r} = G_0$, we mean the entire set of criteria and not a particular criterion or subcriterion; in this particular case, we have $E(G_0) = EL$,
- given $\mathcal{F} \subseteq (\mathcal{G} \setminus EL)$, $E(\mathcal{F}) = \{E(G_\mathbf{r}) : G_\mathbf{r} \in \mathcal{F}\}$, that is $E(\mathcal{F})$ is composed by all elementary subcriteria descending from at least one criterion in $\mathcal{F}$,
- given $G_\mathbf{r} \in \mathcal{G}$, $\mathbf{r} \in \mathcal{I}_\mathcal{G} \cap \mathbb{N}^h$ ($G_\mathbf{r}$ is a criterion at the level h), $1 \leq h < l$, and $k \in \{h+1, \ldots, l\}$, we define:

$$\mathcal{G}^k_\mathbf{r} = \left\{G_{(\mathbf{r},w)} \in \mathcal{G} : (\mathbf{r}, w) \in \mathcal{I}_\mathcal{G} \cap \mathbb{N}^k\right\}$$

  being the set of all subcriteria of criterion $G_\mathbf{r}$ at the level k. (For example, in Figure 1, we have that

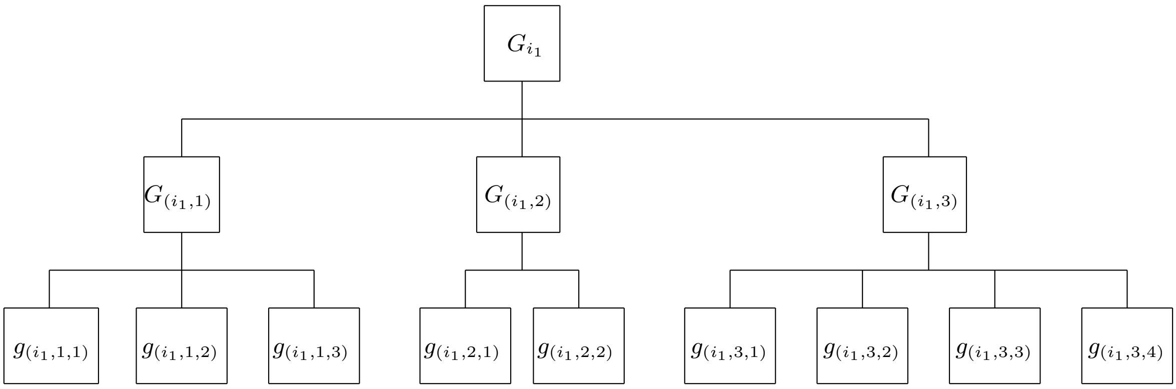
$$\mathcal{G}^2_{\mathbf{i_1}} = \left\{G_{(\mathbf{i_1},1)}, G_{(\mathbf{i_1},2)}, G_{(\mathbf{i_1},3)}\right\} \quad \text{and} \quad \mathcal{G}^3_{(\mathbf{i_1},2)} = \left\{g_{(\mathbf{i_1},2,1)}, g_{(\mathbf{i_2},2,2)}\right\}\Big)$$

Each alternative $a \in A$ is evaluated directly on the elementary subcriteria only, such that to each alternative $a \in A$ there corresponds a vector of evaluations:

$$(g_{\mathbf{t_1}}(a), \ldots, g_{\mathbf{t}_n}(a)), \qquad n = |EL|.$$

Within MCHP, in each node $G_\mathbf{r} \in \mathcal{G}$ of the hierarchy tree there exists a preference relation $\succsim_\mathbf{r}$ on A, such that for all $a, b \in A$, $a \succsim_\mathbf{r} b$ means "a is at least as good as b on subcriterion $G_\mathbf{r}$". In the particular case where $G_\mathbf{r} = g_\mathbf{t}, \mathbf{t} \in EL$, $a \succsim_\mathbf{t} b$ holds if $g_\mathbf{t}(a) \geq g_\mathbf{t}(b)$.

Fig. 1. Hierarchy of criteria for the first level (root) criterion G_{i_1}



4 Multiple Criteria Hierarchy Process for Choquet integral preference model

In this article, we will aggregate the evaluations of alternative $a \in A$ with respect to the elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in EL$, using a Choquet integral as follows.

On the basis of a capacity μ defined on the power set of EL, for all $a, b \in A$, $a \succsim b$ if $C_\mu(a) \geq C_\mu(b)$ where $C_\mu(a)$ and $C_\mu(b)$ are the Choquet integrals with respect to μ of the vectors $[g_{\mathbf{t}}(a), t \in EL]$ and $[g_{\mathbf{t}}(b), t \in EL]$, respectively.
For all $G_{\mathbf{r}} \in \mathcal{G}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \cap \mathbb{N}^h$ ($G_{\mathbf{r}}$ is a criterion at the level h), $h = 1, \ldots, l-1$ and for all $k = h+1, \ldots, l$, we can define the following capacity:

$$\mu_{\mathbf{r}}^k : 2^{\mathcal{G}_{\mathbf{r}}^k} \rightarrow [0,1]$$

such that, for all $\mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^k$, we have that

$$\mu_{\mathbf{r}}^k(\mathcal{F}) = \frac{\mu(E(\mathcal{F}))}{\mu(E(G_{\mathbf{r}}))}$$

In this way, $\mu_{\mathbf{r}}^k$ is a capacity defined on the power set of $\mathcal{G}_{\mathbf{r}}^k$ that could be computed using the capacity μ defined on the power set of EL.
In the following, we shall write $\mu_{\mathbf{r}}$ instead of $\mu_{\mathbf{r}}^l$.

For all $a, b \in A$, $a \succsim_{\mathbf{r}} b$ if $C_{\mu_{\mathbf{r}}}(a) \geq C_{\mu_{\mathbf{r}}}(b)$, where $C_{\mu_{\mathbf{r}}}(a)$ and $C_{\mu_{\mathbf{r}}}(b)$ are the Choquet integrals with respect to $\mu_{\mathbf{r}}$ of the vectors $[g_{\mathbf{t}}(a), t \in E(G_{\mathbf{r}})]$ and $[g_{\mathbf{t}}(b), t \in E(G_{\mathbf{r}})]$, respectively. Observe that for all $a \in A$,

$$C_{\mu_{\mathbf{r}}}(a) = \frac{C_\mu(a_{\mathbf{r}})}{\mu(E(G_{\mathbf{r}}))} \tag{7}$$

where $a_{\mathbf{r}}$ is a fictitious alternative having the same evaluations of a on elementary criteria from $E(G_{\mathbf{r}})$ and null evaluation on criteria outside $E(G_{\mathbf{r}})$, i.e. $g_{\mathbf{s}}(a_{\mathbf{r}}) = g_{\mathbf{s}}(a)$ if $\mathbf{s} \in E(G_{\mathbf{r}})$ and $g_{\mathbf{s}}(a_{\mathbf{r}}) = 0$ if $\mathbf{s} \notin E(G_{\mathbf{r}})$.

The Shapley value expressing the importance of criterion $G_{(\mathbf{r},w)} \in \mathcal{G}_{\mathbf{r}}^k$ being thus a subcriterion of $G_{\mathbf{r}}$ at the level k is:

$$\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w)}) = \sum_{T \subseteq \mathcal{G}_{\mathbf{r}}^k \setminus \{G_{(\mathbf{r},w)}\}} \frac{(|\mathcal{G}_{\mathbf{r}}^k \setminus T| - 1)!|T|!}{|\mathcal{G}_{\mathbf{r}}^k|!} \left[\mu_{\mathbf{r}}^k(T \cup \{G_{(\mathbf{r},w)}\}) - \mu_{\mathbf{r}}^k(T)\right] \quad (8)$$

while the interaction index expressing the sign and the magnitude of the sinergy in a couple of criteria $G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)} \in \mathcal{G}_{\mathbf{r}}^k$ is given by:

$$\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}) = \sum_{T \subseteq \mathcal{G}_{\mathbf{r}}^k \setminus \{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\}} \frac{(|\mathcal{G}_{\mathbf{r}}^k \setminus T| - 2)!|T|!}{(|\mathcal{G}_{\mathbf{r}}^k| - 1)!} \cdot \quad (9)$$

$$\cdot \left[\mu_{\mathbf{r}}^k(T \cup \{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\}) - \mu_{\mathbf{r}}^k(T \cup \{G_{(\mathbf{r},w_1)}\}) - \mu_{\mathbf{r}}^k(T \cup \{G_{(\mathbf{r},w_2)}\}) + \mu_{\mathbf{r}}^k(T)\right]$$

In case the capacity μ on $\{g_{\mathbf{t}}, \mathbf{t} \in EL\}$ is 2-additive, the Shapley value $\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w)})$ and the interaction index $\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)})$, with $G_{(\mathbf{r},w)}, G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)} \in \mathcal{G}_{\mathbf{r}}^k$, can be expressed as follows:

$$\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w)}) =$$

$$\left\{\sum_{\mathbf{t} \in E(G_{(\mathbf{r},w)})} a(g_{\mathbf{t}}) + \sum_{\mathbf{t_1},\mathbf{t_2} \in E(G_{(\mathbf{r},w)})} a(g_{\mathbf{t_1}}, g_{\mathbf{t_2}}) + \sum_{\substack{\mathbf{t_1} \in E(G_{(\mathbf{r},w)}) \\ \mathbf{t_2} \in \mathcal{G}_{\mathbf{r}}^k \setminus \{G_{(\mathbf{r},w)}\}}} \frac{a(g_{\mathbf{t_1}}, g_{\mathbf{t_2}})}{2}\right\} \cdot \frac{1}{\mu(E(G_{\mathbf{r}}))} \quad (10)$$

$$\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}) = \left\{\sum_{\substack{\mathbf{t_1} \in E(G_{(\mathbf{r},w_1)}), \\ \mathbf{t_2} \in E(G_{(\mathbf{r},w_2)})}} a(g_{\mathbf{t_1}}, g_{\mathbf{t_2}})\right\} \cdot \frac{1}{\mu(E(G_{\mathbf{r}}))} \quad (11)$$

Taking into account the expression of the Shapley index in equation (8) and $G_{\mathbf{s}_1}, G_{\mathbf{s}_2} \in G_{\mathbf{r}_1}^k \cap G_{\mathbf{r}_2}^k$ (that is $G_{\mathbf{s}_1}$ and $G_{\mathbf{s}_2}$ are subcriteria of both $G_{\mathbf{r}_1}$ and $G_{\mathbf{r}_2}$ and they are sited at the level k), and supposing, without loss of generality, that $\mathbf{r}_2 = (\mathbf{r}_1, w)$ (that is $G_{\mathbf{r}_2}$ is a subcriterion of $G_{\mathbf{r}_1}$), it is worth noting that the following inequalities could be verified:

$$\varphi_{\mathbf{r}_1}^k(G_{\mathbf{s}_1}) > \varphi_{\mathbf{r}_1}^k(G_{\mathbf{s}_2}) \quad \text{and} \quad \varphi_{\mathbf{r}_2}^k(G_{\mathbf{s}_1}) < \varphi_{\mathbf{r}_2}^k(G_{\mathbf{s}_2}) \quad \text{(or viceversa)}$$

This means that the importance of the criterion $G_{\mathbf{s}_1}$ is greater than the importance of the criterion $G_{\mathbf{s}_2}$ if they are considered as subcriteria of $G_{\mathbf{r}_1}$, but the importance of $G_{\mathbf{s}_2}$ is greater than importance of $G_{\mathbf{s}_1}$ if they are considered as subcriteria of $G_{\mathbf{r}_2}$. We shall show this possibility in the didactic example presented in the next section.

5 A didactic example

Let us consider a set of seven students $A = \{a, b, c, d, e, f, g\}$ evaluated on the basis of two macro subjects: Science and Humanities. Science has two sub-subjects: Mathematics and Physics, while Humanities has two sub-subjects: Literature and Philosophy. The number of levels considered is two.

In terms of notation, we have $\mathcal{G} = \left\{G_{\mathbf{1}}, G_{\mathbf{2}}, G_{(\mathbf{1},1)}, G_{(\mathbf{1},2)}, G_{(\mathbf{2},1)}, G_{(\mathbf{2},2)}\right\}$, and the elements of $\mathcal{G}$ denote respectively, Science, Humanities, Mathematics, Physics, Literature and Philosophy. The students are evaluated on the basis of the elementary criteria only; such evaluations are shown in Table 1.

Table 1. Matrix evaluation

	Science		**Humanities**	
Students	Mathematics	Physics	Literature	Philosophy
a	18	18	12	12
b	16	16	16	16
c	14	14	18	18
d	18	12	16	16
e	15	15	18	14
f	18	14	14	18
g	15	17	18	16

In the following, we shall consider a 2-additive capacity determined by the Möbius measures in Table 2.

Table 2. Möbius measures

$a(G_{(\mathbf{1},1)})$	0.5
$a(G_{(\mathbf{1},2)})$	0.5
$a(G_{(\mathbf{2},1)})$	0.2
$a(G_{(\mathbf{2},2)})$	0.15
$a(G_{(\mathbf{1},1)}, G_{(\mathbf{1},2)})$	-0.45
$a(G_{(\mathbf{1},1)}, G_{(\mathbf{2},1)})$	0
$a(G_{(\mathbf{1},1)}, G_{(\mathbf{2},2)})$	0.1
$a(G_{(\mathbf{1},2)}, G_{(\mathbf{2},1)})$	0.05
$a(G_{(\mathbf{1},2)}, G_{(\mathbf{2},2)})$	0.1
$a(G_{(\mathbf{2},1)}, G_{(\mathbf{2},2)})$	-0.15

Applying the expression (7) of the hierarchal Choquet integral introduced in Section 4, we can compute the evaluation of every student with respect to each subject Science ($G_{\mathbf{1}}$) and Humanities ($G_{\mathbf{2}}$) (see Table 3).

For example, $C_{\mu_{\mathbf{1}}}(a)$ is evaluated by considering a_1 a fictitious alternative with the same evaluations of a on the elementary criteria $E(G_{\mathbf{1}})$ and null evaluations on $E(G_{\mathbf{2}})$. As a result the Choquet integral $C_{\mu_{\mathbf{1}}}(a)$ is given by $\dfrac{C_{\mu}(a_1)}{\mu(E(G_{\mathbf{1}}))}$, where $\mu_{\mathbf{1}}(\mathcal{F}) = \dfrac{\mu(E(\mathcal{F}))}{\mu(E(G_{\mathbf{1}}))}$ with $\mathcal{F} \subseteq G_{\mathbf{1}}$.

Table 3. Choquet integrals with respect to the macro-subjects Science and Humanities

	Science		**Humanities**		
	Mathematics	Physics	Literature	Philosophy	C_{μ_r}
$C_{\mu_{\mathbf{1}}}(a)$	18	18	0	0	18
$C_{\mu_{\mathbf{2}}}(a)$	0	0	12	12	12
$C_{\mu_{\mathbf{1}}}(b)$	16	16	0	0	16
$C_{\mu_{\mathbf{2}}}(b)$	0	0	16	16	16
$C_{\mu_{\mathbf{1}}}(c)$	14	14	0	0	14
$C_{\mu_{\mathbf{2}}}(c)$	0	0	18	18	18
$C_{\mu_{\mathbf{1}}}(d)$	18	12	0	0	17.45
$C_{\mu_{\mathbf{2}}}(d)$	0	0	16	16	16
$C_{\mu_{\mathbf{1}}}(e)$	15	15	0	0	15
$C_{\mu_{\mathbf{2}}}(e)$	0	0	18	14	18
$C_{\mu_{\mathbf{1}}}(f)$	18	14	0	0	17.64
$C_{\mu_{\mathbf{2}}}(f)$	0	0	14	18	17
$C_{\mu_{\mathbf{1}}}(g)$	15	17	0	0	16.82
$C_{\mu_{\mathbf{2}}}(g)$	0	0	18	16	18

By considering the capacities on the elementary criteria displayed in Table 2 and adopting the expression (10) defined in Section 4, we compute the Shapley values of the elementary criteria $G_{(\mathbf{r},i)}$ with respect to their relative overcriterion $G_{\mathbf{r}}$ (see Table 4). Then the overall Shapley values of the elementary criteria (i.e. with respect to $G_{\mathbf{0}}$) are calculated and showed in Table 5. Finally, the Shapley values of subcriteria $G_{\mathbf{1}}$ (Science) and $G_{\mathbf{2}}$ (Humanities) and their interaction index (see the expression (11) introduced in Section 4) are computed and displayed in Table 6.

As it has been pointed out in Section 4, in this example it results that Literature is more important than Philosophy, if they are considered as subcriteria of Humanities (see Table 4); on the contrary Philosophy is more important than Literature if they are considered as subcriteria of the whole set of criteria $G_{\mathbf{0}}$ (see Table 5).

Table 4. Shapley values of every elementary criteria with respect to every macro subject $G_{\mathbf{r}}$

	Science		**Humanities**	
	Mathematics	Physics	Literature	Philosophy
$\varphi^k_{\mathbf{r}}(G_{(\mathbf{r},w)})$	0.5	0.5	0.625	0.375

Table 5. Shapley values of the elementary criteria

	$\varphi^k_{\mathbf{r}}(G_{(\mathbf{r},w)})$
Mathematics	0.325
Physics	0.35
Literature	0.15
Philosophy	0.175

Table 6. The Shapley values and interaction index of Science (G_1) and Humanities (G_2)

	$\varphi^k_{\mathbf{r}}(G_{(\mathbf{r},w)})$
Science	0.675
Humanities	0.325
	$\varphi^k_{\mathbf{r}}(G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)})$
Science and **Humanities**	0.25

6 Conclusions

We have proposed the application of Multiple Criteria Hierarchy Process (MCHP) to a preference model expressed in terms of Choquet integral, in order to deal with interaction between criteria. Application of MCHP to Chqouet integral permits to define importance and interactions of criteria with respect to any subriterion in the hierarchy. To apply MCHP to Choquet integral in real world problems, it is necessary to elicit preference parameters, which, in this case, are the non interactive weights represented by a capacity. MCHP in this context is important because it permits the DM to give preference information related to any criterion in the hierarchy. For example, the DM can say that student a is globally preferred to student b, but he can also say that student c is better than student d in Humanities. DM can also say that criterion Science is more important than Humanities or that the interaction between Physics and Philosophy is greater than the interaction between Mathematics and Literature. Many multicriteria disaggregation procedures have been proposed to infer a capacity from those types of preference information in case the hierarchy of criteria is not considered (see for example, [11]). Recently, a new multicriteria disaggregation method has been proposed to take into account that, in general, more than one capacity is able to represent the preference expressed by the DM: Non Additive Robust Ordinal Regression (NAROR) [1]). NAROR considers all the capacities that are compatible with the preference information given by the DM, adopting the concepts of possible and necessary preference introduced in [9]. In simple words, a is necessarily or possibly preferred to b, if it is preferred for all compatible capacities or for at least one compatible capacity, respectively. In our opinion, application of NAROR to MCHP for Choquet integral will permit to take into account interaction of criteria and hierarchy in a very efficent way in many complex real world problems. Thus, we plan to develop such an extension of NAROR to MCHP applied to Choquet integral in a future paper.

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