Multiple Criteria Hierarchy Process in Robust Ordinal Regression

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Abstract: A great majority of methods designed for Multiple Criteria Decision Aiding (MCDA) assume that all evaluation criteria are considered at the same level, however, it is often the case that a practical application is imposing a hierarchical structure of criteria. The hierarchy helps decomposing complex decision making problems into smaller and manageable subtasks, and thus, it is very attractive for users. To handle the hierarchy of criteria in MCDA, we propose a methodology called Multiple Criteria Hierarchy Process (MCHP) which permits consideration of preference relations with respect to a subset of criteria at any level of the hierarchy. MCHP can be applied to any MCDA method. In this paper, we apply MCHP to Robust Ordinal Regression (ROR) being a family of MCDA methods that takes into account all sets of parameters of an assumed preference model, which are compatible with preference information elicited by a Decision Maker (DM). As a result of ROR, one gets necessary and possible preference relations in the set of alternatives, which hold for all compatible sets of parameters or for at least one compatible set of parameters, respectively. Applying MCHP to ROR one gets to know not only necessary and possible preference relations with respect to the whole set of criteria, but also necessary and possible preference relations related to subsets of criteria at different levels of the hierarchy. We also show how MCHP can be extended to handle group decision and interactions among criteria.

Keywords: Multiple Criteria Decision Aiding, Hierarchy of criteria, Multiple Criteria Hierarchy Process, Robust Ordinal Regression, Preference modeling

1 Introduction

It is well known that the dominance relation established in the set of alternatives evaluated on multiple criteria is the only objective information that comes out from a formulation of a multiple criteria decision

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problem (including sorting, ranking and choice). While dominance relation permits to eliminate many irrelevant (i.e. dominated) alternatives, it does not compare completely all of them, resulting in a situation where many alternatives remain incomparable. This situation may be addressed by taking into account preferences of a Decision Maker (DM). Therefore, all Multiple Criteria Decision Aiding (MCDA) methods (for state-of-the-art surveys on MCDA see [7]) require some preference information elicited by a DM. Information provided by a DM is used within a MCDA process to build a preference model which is then applied on a non-dominated (Pareto-optimal) set of alternatives to arrive at a recommendation.

A great majority of methods designed for MCDA, assume that all evaluation criteria are considered at the same level, however, it is often the case that a practical application is imposing a hierarchical structure of criteria. For example, in economic ranking, alternatives may be evaluated on indicators which aggregate evaluations on several sub-indicators, and these sub-indicators may aggregate another set of sub-indicators, etc. In this case, the marginal value functions may refer to all levels of the hierarchy, representing values of particular scores of the alternatives on indicators, sub-indicators, sub-sub-indicators, etc. Considering hierarchical, instead of flat, structure of criteria, permits decomposition of a complex decision problem into smaller problems involving less criteria. To handle the hierarchy of criteria, we introduce in this paper a Multiple Criteria Hierarchy Process (MCHP). The basic idea of MCHP relies on consideration of preference relations at each node of the hierarchy tree of criteria. These preference relations concern both the phase of eliciting preference information, and the phase of analyzing a final recommendation by the DM. Let us consider a very simple and well known preference model, the linear value function, which assigns to each alternative $a \in A$ the value $U(a) = w_1 g_1(a) + \ldots + w_n g_n(a)$, $w_i \geq 0, i = 1, \ldots n$, where $g_i(a)$ is an evaluation of alternative a on criterion $g_i, i = 1, \ldots, n$. If in the phase of eliciting preference information, the DM declares that alternative a is preferred to alternative b with respect to a criterion which, in a node of the hierarchy tree, groups a set of sub-criteria $\mathcal{G}_{\mathbf{r}}$, this can be modeled as

$$\sum_{i \in \mathcal{G}_{\mathbf{r}}} w_i g_i(a) > \sum_{i \in \mathcal{G}_{\mathbf{r}}} w_i g_i(b),$$

which puts some constraints on the values of admissible weights w_i. In the phase of analyzing a final recommendation, even more important, MCHP shows preference relations $\succsim_{\mathbf{r}}$ on A with respect to the set of subcriteria $\mathcal{G}_{\mathbf{r}}$, such that, for all $a, b \in A$,

$$a \succsim_{\mathbf{r}} b \Leftrightarrow \sum_{i \in \mathcal{G}_{\mathbf{r}}} w_i g_i(a) \geq \sum_{i \in \mathcal{G}_{\mathbf{r}}} w_i g_i(b),$$

where $a \succsim_{\mathbf{r}} b$ reads alternative a is at least as good as alternative b on the set of subcriteria $\mathcal{G}_{\mathbf{r}}$. Analyzing the preference relation $\succsim_{\mathbf{r}}$ is very useful in any decision aiding process because it permits to look into structural elements of the overall preference relation $\succsim$ taking into account the whole set of criteria, and justify better the final recommendation. For example, in a decision problem related to evaluation of students, one can say not only that student a is comprehensively preferred to student b, i.e. $a \succ b$ (where $\succ$ is the asymmetric part of $\succsim$; analogously, in the following, $\succ_{\mathbf{r}}$ is the asymmetric part of $\succsim_{\mathbf{r}}$), but also that a is comprehensively preferred to b because a is preferred to b on subsets of subjects (subcriteria) related to Mathematics and Physics, i.e. $a \succ_{Mathematics} b$ and $a \succ_{Physics} b$, even if b is preferred to a on subjects related to Humanities, i.e. $b \succ_{Humanities} a$. Moreover, one can also say that, for example, a is preferred to b on the subset of subjects related to Mathematics because, considering Analysis and Algebra as subjects (sub-criteria) related to Mathematics, a is preferred to b on Analysis, i.e. $a \succ_{Analysis} b$, and this is enough to compensate the fact that b is preferred to a on Algebra, i.e. $b \succ_{Algebra} a$. Since partial preference relations $\succsim_{Mathematics}$, $\succsim_{Physics}$, $\succsim_{Humanities}$, $\succsim_{Analysis}$, $\succsim_{Algebra}$, and so on, can be constructed using any MCDA methodology, this shows the universal character of MCHP.

In this paper, in order to show the useful features of MCHP, we apply this methodology to a recently proposed family of MCDA methods, called Robust Ordinal Regression (ROR) ([12],[8],[14],[16]). Basic ideas of ROR can be summarized as follows. To deal with a multiple criteria decision problem, Multiple Attribute Utility Theory (MAUT) ([20]) constructs a value function which assigns to each alternative a real number representing its degree of preferability. The first MCDA methods using the ordinal regression approach ([4],[25],[28]), aimed at finding one value function compatible with preference information provided by the DM (see, e.g., [17],[24],[22],[6]). Most frequently additive value functions have been considered, i.e. functions obtained by summing up marginal value functions corresponding to particular criteria. For example, in [17], each marginal value function is a piecewise-linear one. Remark that in case of ordinal regression the preference information is always indirect.

In ordinal regression, and also in ROR, the preference information elicited by the DM is indirect, i.e. the DM provides decision examples, like preferential pairwise comparisons of some selected alternatives. This type of preference information is opposed to the direct one, which is composed of values of parameters of the assumed preference model, like weights or trade-off rates of the weighted sum model. Research indicates that indirect preference elicitation requires less cognitive effort from the DM than the direct one, and thus, it becomes more and more popular.

When building, via ordinal regression, a value function compatible with indirect preference information given as pairwise comparisons of some selected alternatives, one encounters a problem of plurality of compatible value functions. Until recently, the usual practice was to select only one of the compatible value functions, either by the DM or using some mathematical tools for finding a "central" value function. In general, however, each compatible value function gives a different ranking of the considered set of alternatives, and thus, it is reasonable to investigate what is the consequence of applying all compatible value functions on the whole set of considered alternatives. For this reason, ROR takes into account all compatible value functions simultaneously. In this context, two preference relations are considered:

- possible preference relation, for which alternative a is possibly preferred to alternative b if a is at least as good as b for at least one compatible value function, and

- necessary preference relation, for which alternative a is necessarily preferred to alternative b if a is at least as good as b for all compatible value functions.

The first method that applied the concept of ROR was UTA^{GMS} [12]: it takes into account pairwise comparisons of alternatives provided by a DM; GRIP [8] was its generalization taking into account not only pairwise comparisons, but also intensities of preference; ROR has been also applied to sorting problems [14], and it has been adapted to other preference models, like outranking relation [11],[18] and non additive integrals [1].

Applying MCHP to ROR, permits to consider preference information at each level of the hierarchy in the phase of eliciting preference information. Moreover, putting together MCHP and ROR, permits to define necessary and possible preference relations at each node of the hierarchy tree. This gives an insight into evolution of the necessary and possible preference relations along the hierarchy tree. In fact, if we know that a is not necessarily comprehensively preferred to b, with MCHP we can find at which level a particular subcriterion opposes to the conclusion that a is necessarily preferred to b. All the properties that hold for the "flat" version of ROR methods are also valid in the hierarchical context, and other properties that are characteristic to the hierarchical context are given in this paper.

The paper is structured in this way: section 2 describes some basic concepts of the MCHP; section 3 describes the GRIP method adapted to the hierarchical context; in section 4 we present the properties of necessary and possible preference relations; section 5 describes the concept of intensity of preference and most representative value function; in section 6 we present a didactic example; in section 7 we present some extensions of the hierarchical ROR; section 8 ends the paper with conclusions.

2 Multiple Criteria Hierarchy Process (MCHP)

In MCHP, we consider a set $\mathcal{G}$ of hierarchically ordered criteria, i.e. all criteria are not considered at the same level, but they are distributed over l different levels (see Figure 1). At level 1, there are first level criteria called root criteria. Each root criterion has its own hierarchy tree. The leafs of each hierarchy tree are at the last level l and they are called elementary subcriteria. Thus, in graph theory terms, the whole hierarchy is a forest. We will use the following notation:

- $A = \{a, b, c, \ldots\}$ is the finite set of alternatives,
- l is the number of levels in the hierarchy of criteria,
- $\mathcal{G}$ is the set of all criteria at all considered levels,
- $\mathcal{I}_{\mathcal{G}}$ is the set of indices of particular criteria representing position of criteria in the hierarchy,
- m is the number of the first level criteria, $G_1, \ldots, G_m$,
- $G_{\mathbf{r}} \in \mathcal{G}$, with $\mathbf{r} = (i_1, \ldots, i_h) \in \mathcal{I}_{\mathcal{G}}$, denotes a subcriterion of the first level criterion G_{i_1} at level h; the first level criteria are denoted by G_{i_1}, $i_1 = 1, \ldots, m$,
- $n(\mathbf{r})$ is the number of subcriteria of $G_{\mathbf{r}}$ in the subsequent level, i.e. the direct subcriteria of $G_{\mathbf{r}}$ are $G_{(\mathbf{r},1)}, \ldots, G_{(\mathbf{r},n(\mathbf{r}))}$,
- $g_{\mathbf{t}} : A \rightarrow \mathbb{R}$, with $\mathbf{t} = (i_1, \ldots, i_l) \in \mathcal{I}_{\mathcal{G}}$, denotes an elementary subcriterion of the first level criterion G_{i_1}, i.e a criterion at level l of the hierarchy tree of G_{i_1},
- EL is the set of indices of all elementary subcriteria:

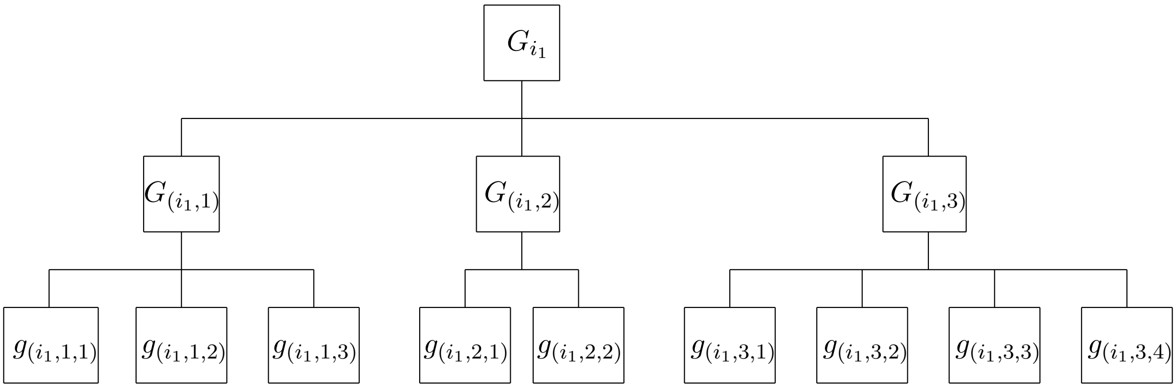
$$EL = \{\mathbf{t} = (i_1, \ldots, i_l) \in \mathcal{I}_{\mathcal{G}}\} \quad \text{where} \begin{cases} i_1 = 1, \ldots, m \\ i_2 = 1, \ldots, n(i_1) \\ \ldots\ldots \\ i_l = 1, \ldots, n(i_1, \ldots, i_{l-1}) \end{cases}$$

- $E(G_{\mathbf{r}})$ is the set of indices of elementary subcriteria descending from $G_{\mathbf{r}}$, i.e.

$$E(G_{\mathbf{r}}) = \{(\mathbf{r}, i_{h+1}, \ldots, i_l) \in \mathcal{I}_{\mathcal{G}}\} \quad \text{where} \begin{cases} i_{h+1} = 1, \ldots, n(\mathbf{r}) \\ \ldots\ldots \\ i_l = 1, \ldots, n(\mathbf{r}, i_{h+1}, \ldots, i_{l-1}) \end{cases}$$

  thus, $E(G_{\mathbf{r}}) \subseteq EL$,
- when $\mathbf{r} = 0$, then by $G_{\mathbf{r}} = G_0$, we mean the entire set of criteria and not a particular criterion or subcriterion; in this particular case, we have $E(G_0) = EL$.

Figure 1: Hierarchy of criteria for the first level (root) criterion G_{i_1}



Without loss of generality we suppose that each elementary subcriterion $g_{\mathbf{t}}, \mathbf{t} \in EL$, maps alternatives to real numbers $g_{\mathbf{t}} : A \rightarrow \mathbb{R}$, such that for all $a, b \in A$, $g_{\mathbf{t}}(a) \geq g_{\mathbf{t}}(b)$ means that a is at least as good as b with respect to elementary criterion $g_{\mathbf{t}}$. If criterion $g_{\mathbf{t}}$ has, originally, an ordered qualitative scale, e.g., very bad, bad, medium, good, very good, one can number code such linguistic labels in a way maintaining the preference order. Each alternative $a \in A$ is evaluated directly on the elementary subcriteria only, such that to each alternative $a \in A$ there corresponds a vector of evaluations:

$$\left(g_{\mathbf{t_1}}(a), \ldots, g_{\mathbf{t}_n}(a)\right), \qquad n = |EL|\,.$$

Within MCHP, in each node $G_{\mathbf{r}} \in \mathcal{G}$ of the hierarchy tree there exists a preference relation $\succsim_{\mathbf{r}}$ on A, such that for all $a, b \in A$, $a \succsim_{\mathbf{r}} b$ means "a is at least as good as b on subcriterion $G_{\mathbf{r}}$". In the particular case where $G_{\mathbf{r}} = g_{\mathbf{t}}, \mathbf{t} \in EL$, $a \succsim_{\mathbf{t}} b$ holds if $g_{\mathbf{t}}(a) \geq g_{\mathbf{t}}(b)$.

A minimal requirement that preference relations $\succsim_{\mathbf{r}}$ have to satisfy is a dominance principle for hierarchy of criteria, stating that if alternative a is at least as good as alternative b for all subcriteria $G_{(\mathbf{r},j)}$ of $G_{\mathbf{r}}$ of the level immediately below, then a is at least as good as b on $G_{\mathbf{r}}$. For example, if student a is at least as good as student b on Algebra and Analysis, being subcriteria of Mathematics, then a is at least as good as b on Mathematics. Formally, this dominance principle can be stated as follows: given $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, if $a \succsim_{(\mathbf{r},j)} b$ for all $j = 1, \ldots, n(\mathbf{r})$ then $a \succsim_{\mathbf{r}} b$.

In this article, we will aggregate the evaluations of alternative $a \in A$ with respect to the elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in EL$, using an additive value function:

$$U(g_{\mathbf{t_1}}(a), \ldots, g_{\mathbf{t}_n}(a)) = \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(g_{\mathbf{t}}(a)), \tag{1}$$

where $u_{\mathbf{t}}$ are marginal value functions, non-decreasing with respect to the evaluation expressed by its argument. Analogously, the marginal value function of alternative $a \in A$ on criterion $G_{\mathbf{r}} \in \mathcal{G}$, is given by:

$$U_{\mathbf{r}}(g_{\mathbf{t}}(a), \mathbf{t} \in E(G_{\mathbf{r}})) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(g_{\mathbf{t}}(a)), \tag{2}$$

such that for all $a, b \in A$, $a \succsim_{\mathbf{r}} b$ iff $U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b)$.

In the following, to simplify the notation, we shall write $U(a)$ instead of $U(g_{\mathbf{t_1}}(a), \ldots, g_{\mathbf{t}_n}(a))$, $U_{\mathbf{r}}(a)$ instead of $U_{\mathbf{r}}(g_{\mathbf{t}}(a), \mathbf{t} \in E(G_{\mathbf{r}}))$, and $u_{\mathbf{t}}(a)$ instead of $u_{\mathbf{t}}(g_{\mathbf{t}}(a))$.

3 Multiple Criteria Hierarchy Process applied to a Robust Ordinal Regression method

When aggregating evaluations of alternatives on multiple elementary subcriteria, we will take into account some preference information provided by the DM. This preference information concerns a subset of alternatives $A^R \subseteq A$, called reference alternatives, on which the DM is relatively more confident than on the others. The DM is expected to provide the following preference information:

- a partial preorder $\succsim$ on A^R, whose meaning is: for $a^*, b^* \in A^R$

  $$a^* \succsim b^* \;\Leftrightarrow\; \text{“ } a^* \text{ is at least as good as } b^* \text{ ”}.$$

  Denoting by $\succsim^{-1}$ the inverse of $\succsim$, i.e. if $a^* \succsim b^*$ then $b^* \succsim^{-1} a^*$, $\sim$ (indifference) is the symmetric part of $\succsim$ given by $\succsim \cap \succsim^{-1}$, i.e. if $a^* \sim b^*$ then $a^* \succsim b^*$ and $a^* \succsim^{-1} b^*$, and $\succ$ (preference) is the asymmetric part given by $(\succsim \setminus \sim)$, i.e. if $a^* \succ b^*$ then $a^* \succsim b^*$ and not $a^* \sim b^*$;

- a partial preorder $\succsim^*$ on $A^R \times A^R$, whose meaning is: for $a^*, b^*, c^*, d^* \in A^R$,

  $$(a^*, b^*) \succsim^* (c^*, d^*) \Leftrightarrow \text{“} a^* \text{ is preferred to } b^* \text{ at least as much as } c^* \text{ is preferred to } d^* \text{ ”}.$$

  Analogously to $\succsim$, $\succ^*$ and $\sim^*$ are the asymmetric and the symmetric part of $\succsim^*$;

- given $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$, a partial preorder $\succsim_{\mathbf{r}}$ on A^R, whose meaning is: for $a^*, b^* \in A^R$,

  $$a^* \succsim_{\mathbf{r}} b^* \Leftrightarrow \text{“} a^* \text{ is at least as good as } b^* \text{ with respect to subcriterion } G_{\mathbf{r}}.\text{”}$$

  Analogously to $\succsim$, $\succ_{\mathbf{r}}$ and $\sim_{\mathbf{r}}$ are the asymmetric and the symmetric part of $\succsim_{\mathbf{r}}$;

- given $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$, a partial preorder $\succsim^*_{\mathbf{r}}$ on $A^R \times A^R$, whose meaning is: for $a^*, b^*, c^*, d^* \in A^R$,

$$(a^*, b^*) \succsim^*_{\mathbf{r}} (c^*, d^*) \Leftrightarrow \text{“}a^* \text{ is preferred to } b^* \text{ at least as much as } c^* \text{ is preferred to } d^* \text{ with respect to subcriterion } G_{\mathbf{r}}\text{”}.$$

Analogously to $\succsim$, $\succ^*_{\mathbf{r}}$ and $\sim^*_{\mathbf{r}}$ are the asymmetric and the symmetric part of $\succsim^*_{\mathbf{r}}$.

An additive value function is called *compatible* if it is able to restore the preference information supplied by the DM. Therefore, an additive value function (1) is compatible if it satisfies the following set of linear constraints:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
U(a^*) > U(b^*) \quad \text{if} \;\; a^* \succ b^* \\
U(a^*) = U(b^*) \quad \text{if} \;\; a^* \sim b^* \\
U(a^*) - U(b^*) > U(c^*) - U(d^*) \quad \text{if} \;\; (a^*, b^*) \succ^* (c^*, d^*) \\
U(a^*) - U(b^*) = U(c^*) - U(d^*) \quad \text{if} \;\; (a^*, b^*) \sim^* (c^*, d^*) \\
U_{\mathbf{r}}(a^*) > U_{\mathbf{r}}(b^*) \quad \text{if} \;\; a^* \succ_{\mathbf{r}} b^* \\
U_{\mathbf{r}}(a^*) = U_{\mathbf{r}}(b^*) \quad \text{if} \;\; a^* \sim_{\mathbf{r}} b^* \\
U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) > U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) \quad \text{if} \;\; (a^*, b^*) \succ^*_{\mathbf{r}} (c^*, d^*) \\
U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) = U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) \quad \text{if} \;\; (a^*, b^*) \sim^*_{\mathbf{r}} (c^*, d^*)
\end{array}\right\} a^*, b^*, c^*, d^* \in A^R, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL \\
u_{\mathbf{t}}(x^k_{\mathbf{t}}) - u_{\mathbf{t}}(x^{k-1}_{\mathbf{t}}) \geq 0, \;\; \forall \mathbf{t} \in EL, \; k = 2, ..., m_{\mathbf{t}}(A^R) \\
u_{\mathbf{t}}(x^1_{\mathbf{t}}) \geq 0, \; u_{\mathbf{t}}(x^{m_{\mathbf{t}}(A^R)}_{\mathbf{t}}) \leq u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}), \;\; \forall \mathbf{t} \in EL \\
u_{\mathbf{t}}(x^0_{\mathbf{t}}) = 0, \;\; \forall \mathbf{t} \in EL \\
\sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}) = 1.
\end{array}\right\} \left(E^{A^R}\right)$$

where, $x^0_{\mathbf{t}} = min_{a \in A} \; g_{\mathbf{t}}(a)$, and $x^{m_{\mathbf{t}}}_{\mathbf{t}} = max_{a \in A} \; g_{\mathbf{t}}(a)$; $x^k_{\mathbf{t}} \in X_{\mathbf{t}}(A^R), k = 1, ..., m_{\mathbf{t}}(A^R)$, with $X_{\mathbf{t}}(A^R) \subseteq X_{\mathbf{t}}$, is the set of all different evaluations of reference alternatives from A^R on elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in EL$, and $m_{\mathbf{t}}(A^R) = \left|X_{\mathbf{t}}(A^R)\right|$. The values $x^k_{\mathbf{t}}, k = 1, ..., m_{\mathbf{t}}(A^R)$, are increasingly ordered, i.e.,

$$x^1_{\mathbf{t}} < x^2_{\mathbf{t}} < ... < x^{m_{\mathbf{t}}(A^R)-1}_{\mathbf{t}} < x^{m_{\mathbf{t}}(A^R)}_{\mathbf{t}}.$$

In order to check the existence of a compatible value function, one has to transform first the strict inequalities of E^{A^R} by adding an auxiliary variable ε. Then, we have to solve the following linear programming problem where the variables are the marginal value functions $u_{\mathbf{t}}(x^k_{\mathbf{t}}), \; k = 1, \ldots, m_{\mathbf{t}}(A^R)$, and

$u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}), \mathbf{t} \in EL$, as well as ε :

$$\left.\begin{array}{l}
\text{maximize } \ \varepsilon, \ \text{ subject to the constraints:}\\
\left.\begin{array}{l}
U(a^*) \geq U(b^*) + \varepsilon \quad \text{if } \ a^* \succ b^* \\
U(a^*) = U(b^*) \quad \text{if } \ a^* \sim b^* \\
U(a^*) - U(b^*) \geq U(c^*) - U(d^*) + \varepsilon \quad \text{if } \ (a^*, b^*) \succ^* (c^*, d^*) \\
U(a^*) - U(b^*) = U(c^*) - U(d^*) \quad \text{if } \ (a^*, b^*) \sim^* (c^*, d^*) \\
U_{\mathbf{r}}(a^*) \geq U_{\mathbf{r}}(b^*) + \varepsilon \quad \text{if } \ a^* \succ_{\mathbf{r}} b^* \\
U_{\mathbf{r}}(a^*) = U_{\mathbf{r}}(b^*) \quad \text{if } \ a^* \sim_{\mathbf{r}} b^* \\
U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) \geq U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) + \varepsilon \quad \text{if } \ (a^*, b^*) \succ_{\mathbf{r}}^* (c^*, d^*) \\
U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) = U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) \quad \text{if } \ (a^*, b^*) \sim_{\mathbf{r}}^* (c^*, d^*)
\end{array}\right\} a^*, b^*, c^*, d^* \in A^R, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL \\
u_{\mathbf{t}}(x_{\mathbf{t}}^k) - u_{\mathbf{t}}(x_{\mathbf{t}}^{k-1}) \geq 0, \ \ \forall \mathbf{t} \in EL, \ k = 2, ..., m_{\mathbf{t}}(A^R) \\
u_{\mathbf{t}}(x_{\mathbf{t}}^1) \geq 0, \ u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}(A^R)}) \leq u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}), \ \ \forall \mathbf{t} \in EL \\
u_{\mathbf{t}}(x_{\mathbf{t}}^0) = 0, \ \ \forall \mathbf{t} \in EL \\
\sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) = 1.
\end{array}\right\} \left(E^{A^{R'}}\right)$$

If $\varepsilon(E^{A^{R'}}) > 0$, where $\varepsilon(E^{A^{R'}}) = \max \varepsilon$, s.t. constraints $E^{A^{R'}}$, then there exists at least one compatible value function $U(\cdot)$; if instead $\varepsilon(E^{A^{R'}}) \leq 0$, then there does not exist any compatible value function $U(\cdot)$. Supposing that there exists at least one compatible value function, each of these functions may induce a different ranking on the whole set A; for this reason the ROR methods, (see [12],[8],[14],[11],[18],[1]), do not take into consideration only one compatible value function but all the compatible value functions simultaneously (we shall denote the set of all compatible value functions by $\mathcal{U}$). Application on the ROR involves the following definitions:

Definition 3.1. *Given two alternatives $a, b \in A$, we say that a is weakly necessarily preferred to b, and we write $a \succsim^N b$, if a is at least as good as b for all compatible value functions:*

$$a \succsim^N b \Leftrightarrow U(a) \geq U(b) \ \ \forall U \in \mathcal{U}.$$

Definition 3.2. *Given two alternatives $a, b \in A$, we say that a is weakly possibly preferred to b, and we write $a \succsim^P b$, if a is at least as good as b for at least one compatible value function:*

$$a \succsim^P b \Leftrightarrow \exists U \in \mathcal{U} : \ \ U(a) \geq U(b).$$

Definition 3.3. *Given two alternatives $a, b \in A$, we say that a is weakly necessarily preferred to b with* *respect to subcriterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and we write $a \succsim^{N}_{\mathbf{r}} b$, if a is at least as good as b with respect to subcriterion $G_{\mathbf{r}}$ for all compatible value functions:*

$$a \succsim^{N}_{\mathbf{r}} b \Leftrightarrow U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \;\; \forall U \in \mathcal{U}.$$

Definition 3.4. *Given two alternatives $a, b \in A$, we say that a is weakly possibly preferred to b with respect to criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and we write $a \succsim^{P}_{\mathbf{r}} b$, if a is at least as good as b with respect to criterion $G_{\mathbf{r}}$ for at least one compatible value function:*

$$a \succsim^{P}_{\mathbf{r}} b \Leftrightarrow \exists U \in \mathcal{U}: \;\; U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b).$$

Note that for $\mathbf{r} \in EL$, we have:

$$\succsim^{N}_{\mathbf{r}} = \succsim^{P}_{\mathbf{r}} = \left\{(a,b) \in A \times A : g_{\mathbf{r}}(a) \geq g_{\mathbf{r}}(b)\right\}.$$

Let us remark that we need both the possible and the necessary preference relation $\succsim^{P}$ and $\succsim^{N}$. In fact, considering the necessary preference relation $\succsim^{N}$ only, we loose some important information given by the ROR methodology. For example, for $a, b \in A$, let us consider the two following cases:

case 1) $a \succsim^{N} b$ and $b \succsim^{P} a$,

case 2) $a \succsim^{N} b$ and $b \not\succsim^{P} a$.

In both, case 1) and case 2), $\forall U \in \mathcal{U}$, $U(a) \geq U(b)$. However, in case 1) there is at least one compatible value function $U \in \mathcal{U}$ such that $U(b) \geq U(a)$, while this does not happen in case 2). If we consider only the necessary preference relation $\succsim^{N}$ we are not able to distinguish case 1) from case 2), while, this is not the case if we use also the possible preference relation $\succsim^{P}$.

Necessary weak preference relations ($\succsim^{N}$ and $\succsim^{N}_{\mathbf{r}}$), and possible weak preference relations ($\succsim^{P}$ and $\succsim^{P}_{\mathbf{r}}$) can be calculated as follows. For all alternatives $a, b \in A$, let $X_{\mathbf{t}}(A^{R} \cup \{a,b\}) \subseteq X_{\mathbf{t}}$ be the set of all different evaluations of alternatives from $A^{R} \cup \{a,b\}$ on criterion $g_{\mathbf{t}}, \mathbf{t} \in EL$, and $m_{\mathbf{t}}(A^{R} \cup \{a,b\}) = |X_{\mathbf{t}}(A^{R} \cup \{a,b\})|$. The values $x^{k}_{\mathbf{t}} \in X_{\mathbf{t}}(A^{R} \cup \{a,b\}),\; k = 1, \ldots, m_{\mathbf{t}}(A^{R} \cup \{a,b\})$, are increasingly ordered, i.e.,

$$x^{1}_{\mathbf{t}} < x^{2}_{\mathbf{t}} < \ldots < x^{m_{\mathbf{t}}(A^{R} \cup \{a,b\})-1}_{\mathbf{t}} < x^{m_{\mathbf{t}}(A^{R} \cup \{a,b\})}_{\mathbf{t}}.$$

Then, the characteristic points of $u_{\mathbf{t}}(\cdot)$, $\mathbf{t} \in EL$, are in $x^{0}_{\mathbf{t}},\; x^{k}_{\mathbf{t}},\; k = 1, \ldots, m_{\mathbf{t}}(A^{R} \cup \{a,b\}),\; x^{m_{\mathbf{t}}}_{\mathbf{t}}$.

Let us consider the following ordinal regression constraints $E(a,b)$, with $u_{\mathbf{t}}(x_{\mathbf{t}}^{k}), \mathbf{t} \in EL,\ k=1,\ldots,m_{\mathbf{t}}(A^R \cup \{a,b\}),\ u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}), \mathbf{t} \in EL$, and ε as variables:

$$\left.\begin{array}{l}\left.\begin{array}{l}U(a^*) \geq U(b^*) + \varepsilon \quad \text{if} \quad a^* \succ b^* \\ U(a^*) = U(b^*) \quad \text{if} \quad a^* \sim b^* \\ U(a^*) - U(b^*) \geq U(c^*) - U(d^*) + \varepsilon \quad \text{if} \quad (a^*,b^*) \succ^* (c^*,d^*) \\ U(a^*) - U(b^*) = U(c^*) - U(d^*) \quad \text{if} \quad (a^*,b^*) \sim^* (c^*,d^*) \\ U_{\mathbf{r}}(a^*) \geq U_{\mathbf{r}}(b^*) + \varepsilon \quad \text{if} \quad a^* \succ_{\mathbf{r}} b^* \\ U_{\mathbf{r}}(a^*) = U_{\mathbf{r}}(b^*) \quad \text{if} \quad a^* \sim_{\mathbf{r}} b^* \\ U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) \geq U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) + \varepsilon \quad \text{if} \quad (a^*,b^*) \succ^*_{\mathbf{r}} (c^*,d^*) \\ U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) = U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) \quad \text{if} \quad (a^*,b^*) \sim^*_{\mathbf{r}} (c^*,d^*)\end{array}\right\} a^*,b^*,c^*,d^* \in A^R;\ \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL \\ u_{\mathbf{t}}(x_{\mathbf{t}}^{k}) - u_{\mathbf{t}}(x_{\mathbf{t}}^{k-1}) \geq 0,\ \mathbf{t} \in EL,\ k=2,...,m_{\mathbf{t}}(A^R \cup \{a,b\}) \\ u_{\mathbf{t}}(x_{\mathbf{t}}^{1}) \geq 0,\ u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}(A^R \cup \{a,b\})}) \leq u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}),\ \mathbf{t} \in EL \\ u_{\mathbf{t}}(x_{\mathbf{t}}^{0}) = 0,\ \mathbf{t} \in EL \\ \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) = 1.\end{array}\right\} (E(a,b))$$

The above constraints depend also on the pair of alternatives $a,b \in A$ because their evaluations $g_{\mathbf{t}}(a)$ and $g_{\mathbf{t}}(b)$ give coordinates for two of $m_{\mathbf{t}}(A^R \cup \{a,b\})$ characteristic points of marginal value function $u_{\mathbf{t}}(\cdot)$, for each $\mathbf{t} \in EL$.

For all $a,b \in A$, and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, let us consider the following sets of constraints:

$$\left.\begin{array}{l}U(b) \geq U(a) + \varepsilon \\ E(a,b)\end{array}\right\} (E^N(a,b)), \qquad \left.\begin{array}{l}U(a) \geq U(b) \\ E(a,b)\end{array}\right\} (E^P(a,b)),$$

$$\left.\begin{array}{l}U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(a) + \varepsilon \\ E(a,b)\end{array}\right\} (E^N_{\mathbf{r}}(a,b)), \qquad \left.\begin{array}{l}U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \\ E(a,b)\end{array}\right\} (E^P_{\mathbf{r}}(a,b)).$$

Thus, we get:

- $a \succsim^N b \Leftrightarrow$ if $E^N(a,b)$ is infeasible or $\varepsilon^N(a,b) \leq 0$, where $\varepsilon^N(a,b) = \max \varepsilon$, s.t. constraints $E^N(a,b)$;
- $a \succsim^P b \Leftrightarrow$ if $E^P(a,b)$ is feasible and $\varepsilon^P(a,b) > 0$, where $\varepsilon^P(a,b) = \max \varepsilon$, s.t. constraints $E^P(a,b)$;
- $a \succsim^N_{\mathbf{r}} b \Leftrightarrow$ if $E^N_{\mathbf{r}}(a,b)$ is infeasible or $\varepsilon^N_{\mathbf{r}}(a,b) \leq 0$, where $\varepsilon^N_{\mathbf{r}}(a,b) = \max \varepsilon$, s.t. constraints $E^N_{\mathbf{r}}(a,b)$;
- $a \succsim^P_{\mathbf{r}} b \Leftrightarrow$ if $E^P_{\mathbf{r}}(a,b)$ is feasible and $\varepsilon^P_{\mathbf{r}}(a,b) > 0$, where $\varepsilon^P_{\mathbf{r}}(a,b) = \max \varepsilon$, s.t. constraints $E^P_{\mathbf{r}}(a,b)$.

4 Properties of necessary and possible preference relations

The necessary and possible preference relations satisfy some interesting properties presented in the following propositions:

Proposition 4.1.

- $\succsim^N \subseteq \succsim^P$; *[12]*
- $\succsim^N$ *is a partial preorder (i.e. reflexive and transitive); [12]*
- $\succsim^P$ *is strongly complete (i.e. for all* $a, b \in A$*,* $a \succsim^P b$ *or* $b \succsim^P a$*) and negatively transitive; [12]*
- $a \succsim^N b$ *or* $b \succsim^P a$*,* $\forall a, b \in A$*; [12]*
- $a \succsim^N b$ *and* $b \succsim^P c$*, then* $a \succsim^P c$*,* $\forall a, b, c \in A$*; [8]*
- $a \succsim^P b$ *and* $b \succsim^N c$*, then* $a \succsim^P c$*,* $\forall a, b, c \in A$*. [8]*

In case of the hierarchy of criteria, some further properties hold, as showed by the following proposition.

Proposition 4.2. *For every* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$*,*

1. $\succsim^N_{\mathbf{r}} \subseteq \succsim^P_{\mathbf{r}}$;
2. $\succsim^N_{\mathbf{r}}$ *is a partial preorder (i.e. reflexive and transitive);*
3. $\succsim^P_{\mathbf{r}}$ *is strongly complete (i.e. for all* $a, b \in A$*,* $a \succsim^P_{\mathbf{r}} b$ *or* $b \succsim^P_{\mathbf{r}} a$*) and negatively transitive;*
4. $a \succsim^N_{\mathbf{r}} b$ *or* $b \succsim^P_{\mathbf{r}} a$*,* $\forall a, b \in A$*;*
5. $a \succsim^N_{\mathbf{r}} b$ *and* $b \succsim^P_{\mathbf{r}} c$*, then* $a \succsim^P_{\mathbf{r}} c$*,* $\forall a, b, c \in A$*;*
6. $a \succsim^P_{\mathbf{r}} b$ *and* $b \succsim^N_{\mathbf{r}} c$*, then* $a \succsim^P_{\mathbf{r}} c$*,* $\forall a, b, c \in A$*.*

Proof. See Appendix. □

Let us observe that if we consider the comprehensive preference represented by the value function U at a "zero" level of the hierarchy, where $\mathbf{r} = 0$, we can consider Proposition (4.1) as a specific case of Proposition (4.2), e.g., we can write $\succsim^N_0 \subseteq \succsim^P_0$ instead of $\succsim^N \subseteq \succsim^P$. The next proposition presents some results which are specific for the ROR in case of the hierarchy of criteria.

Proposition 4.3. *For every* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$*,*

1. *given two alternatives* $a, b \in A$,

$$a \succsim^N_{(\mathbf{r},j)} b \;\; \forall j = 1, \ldots, n(\mathbf{r}) \;\Rightarrow\; a \succsim^N_{\mathbf{r}} b;$$

2. *given two alternatives* $a, b \in A$ *such that:*

   α) $a \succsim^N_{(\mathbf{r},j)} b, \quad \forall j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{w\}$

   β) $a \succsim^P_{(\mathbf{r},w)} b$,

   then $a \succsim^P_{\mathbf{r}} b$;

3. *given two alternatives* $a, b \in A$,

$$a \not\succsim^P_{(\mathbf{r},j)} b \;\; \forall j \in \{1, \ldots, n(\mathbf{r})\} \;\Rightarrow\; a \not\succsim^P_{\mathbf{r}} b.$$

Proof. See Appendix. □

5 Intensity of preference and a representative value function

5.1 Intensity of preference

As in the GRIP method [8], also in case of the hierarchy of criteria it is possible to define quaternary relations $\succsim^{*N}$, $\succsim^{*P}$, $\succsim^{*N}_{\mathbf{t}}$ and $\succsim^{*P}_{\mathbf{t}}$, $\mathbf{t} \in EL$, related to intensity of preference, as follows:

- for each $a, b, c, d \in A$, we say that a is necessarily preferred to b at least as strongly as c is preferred to d, and we write $(a, b) \succsim^{*N} (c, d)$, if a is preferred to b at least as strongly as c is preferred to d for all compatible value functions:

$$(a, b) \succsim^{*N} (c, d) \Leftrightarrow U(a) - U(b) \geq U(c) - U(d), \forall U \in \mathcal{U};$$

- for each $a, b, c, d \in A$, we say that a is possibly preferred to b at least as strongly as c is preferred to d, and we write $(a, b) \succsim^{*P} (c, d)$, if a is preferred to b at least as strongly as c is preferred to d for at least one compatible value function:

$$(a, b) \succsim^{*P} (c, d) \Leftrightarrow \exists U \in \mathcal{U} : U(a) - U(b) \geq U(c) - U(d);$$

- for each $a, b, c, d \in A$, we say that a is necessarily preferred to b at least as strongly as c is preferred to d

with respect to elementary subcriterion $g_{\mathbf{t}}$, and we write $(a,b) \succsim^{*N}_{\mathbf{t}} (c,d)$, if a is preferred to b at least as strongly as c is preferred to d with respect to $g_{\mathbf{t}}$ for all compatible value functions:

$$(a,b) \succsim^{*N}_{\mathbf{t}} (c,d) \Leftrightarrow u_{\mathbf{t}}(a) - u_{\mathbf{t}}(b) \geq u_{\mathbf{t}}(c) - u_{\mathbf{t}}(d), \forall U \in \mathcal{U};$$

- for each $a,b,c,d \in A$, we say that a is possibly preferred to b at least as strongly as c is preferred to d with respect to elementary subcriterion $g_{\mathbf{t}}$, and we write $(a,b) \succsim^{*P}_{\mathbf{t}} (c,d)$, if a is preferred to b at least as strongly as c is preferred to d with respect to $g_{\mathbf{t}}$ for at least one compatible value function:

$$(a,b) \succsim^{*P}_{\mathbf{t}} (c,d) \Leftrightarrow \exists U \in \mathcal{U} : u_{\mathbf{t}}(a) - u_{\mathbf{t}}(b) \geq u_{\mathbf{t}}(c) - u_{\mathbf{t}}(d).$$

In case of the hierarchy of criteria, we can further consider quaternary relations $\succsim^{*N}_{\mathbf{r}}$ and $\succsim^{*P}_{\mathbf{r}}$, related to intensity of preference with respect to subcriterion $G_{\mathbf{r}} \in \mathcal{G}$ at an intermediate level of the hierarchy, as follows:

- for each $a,b,c,d \in A$, and for each $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$, we say that a is necessarily preferred to b at least as strongly as c is preferred to d with respect to subcriterion $G_{\mathbf{r}}$, and we write $(a,b) \succsim^{*N}_{\mathbf{r}} (c,d)$, if a is preferred to b at least as strongly as c is preferred to d with respect to subcriterion $G_{\mathbf{r}}$ for all compatible value functions:

$$(a,b) \succsim^{*N}_{\mathbf{r}} (c,d) \Leftrightarrow U_{\mathbf{r}}(a) - U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(c) - U_{\mathbf{r}}(d), \forall U \in \mathcal{U};$$

- for each $a,b,c,d \in A$, and for each $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$, we say that a is possibly preferred to b at least as strongly as c is preferred to d with respect to subcriterion $G_{\mathbf{r}}$, and we write $(a,b) \succsim^{*N}_{\mathbf{r}} (c,d)$, if a is preferred to b at least as strongly as c is preferred to d with respect to subcriterion $G_{\mathbf{r}}$ for at least one compatible value function:

$$(a,b) \succsim^{*P}_{\mathbf{r}} (c,d) \Leftrightarrow \exists U \in \mathcal{U} : U_{\mathbf{r}}(a) - U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(c) - U_{\mathbf{r}}(d).$$

Observe that quaternary relations $\succsim^{*N}_{\mathbf{t}}$ and $\succsim^{*P}_{\mathbf{t}}$, $\mathbf{t} \in EL$, are a particular case of quaternary relations $\succsim^{*N}_{\mathbf{r}}$ and $\succsim^{*P}_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$, in case $\mathbf{r} \in EL$.

Quaternary relations $\succsim^{*N}$ and $\succsim^{*P}$, $\succsim^{*N}_{\mathbf{r}}$ and $\succsim^{*P}_{\mathbf{r}}$, and $\succsim^{*N}_{\mathbf{t}}$ and $\succsim^{*N}_{\mathbf{t}}$ can be computed as follows. For all alternatives $a,b,c,d \in A$, let $X_{\mathbf{t}}(A^R \cup \{a,b,c,d\}) \subseteq X_{\mathbf{t}}$ be the set of all different evaluations of alternatives from $A^R \cup \{a,b,c,d\}$ on elementary subcriterion $g_{\mathbf{t}}$, $\mathbf{t} \in EL$, and $m_{\mathbf{t}}(A^R \cup \{a,b,c,d\}) = |X_{\mathbf{t}}(A^R \cup \{a,b,c,d\})|$. The values $x^k_{\mathbf{t}} \in X_{\mathbf{t}}(A^R \cup \{a,b,c,d\}),\ k = 1, \ldots, m_{\mathbf{t}}(A^R \cup \{a,b,c,d\})$, are increasingly ordered, i.e.,

$$x^1_{\mathbf{t}} < x^2_{\mathbf{t}} < \ldots < x^{m_{\mathbf{t}}(A^R \cup \{a,b,c,d\})-1}_{\mathbf{t}} < x^{m_{\mathbf{t}}(A^R \cup \{a,b,c,d\})}_{\mathbf{t}}.$$

Then, the characteristic points of $u_{\mathbf{t}}(\cdot)$, $\mathbf{t} \in EL$, are in $x_{\mathbf{t}}^0,\ x_{\mathbf{t}}^k,\ k=1,\ldots,m_{\mathbf{t}}(A^R \cup \{a,b,c,d\}),\ x_{\mathbf{t}}^{m_{\mathbf{t}}}$.

Let us consider the following ordinal regression constraints $E(a,b,c,d)$, with $u_{\mathbf{t}}(x_{\mathbf{t}}^k)$, $\mathbf{t} \in EL$, $k = 1,\ldots,m_{\mathbf{t}}(A^R \cup \{a,b,c,d\})$, $u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}), \mathbf{t} \in EL$, and ε as variables:

$$\left.\begin{array}{l}\left.\begin{array}{l}U(a^*) \geq U(b^*) + \varepsilon \quad \text{if} \quad a^* \succ b^* \\ U(a^*) = U(b^*) \quad \text{if} \quad a^* \sim b^* \\ U(a^*) - U(b^*) \geq U(c^*) - U(d^*) + \varepsilon \quad \text{if} \quad (a^*,b^*) \succ^* (c^*,d^*) \\ U(a^*) - U(b^*) = U(c^*) - U(d^*) \quad \text{if} \quad (a^*,b^*) \sim^* (c^*,d^*) \\ U_{\mathbf{r}}(a^*) \geq U_{\mathbf{r}}(b^*) + \varepsilon \quad \text{if} \quad a^* \succ_{\mathbf{r}} b^* \\ U_{\mathbf{r}}(a^*) = U_{\mathbf{r}}(b^*) \quad \text{if} \quad a^* \sim_{\mathbf{r}} b^* \\ U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) \geq U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) + \varepsilon \quad \text{if} \quad (a^*,b^*) \succ_{\mathbf{r}}^* (c^*,d^*) \\ U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) = U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) \quad \text{if} \quad (a^*,b^*) \sim_{\mathbf{r}}^* (c^*,d^*)\end{array}\right\} a^*,b^*,c^*,d^* \in A^R;\ \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL \\ u_{\mathbf{t}}(x_{\mathbf{t}}^k) - u_{\mathbf{t}}(x_{\mathbf{t}}^{k-1}) \geq 0,\ \mathbf{t} \in EL,\ k = 2,...,m_{\mathbf{t}}(A^R \cup \{a,b,c,d\}) \\ u_{\mathbf{t}}(x_{\mathbf{t}}^1) \geq 0,\ u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}(A^R \cup \{a,b,c,d\})}) \leq u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}),\ \mathbf{t} \in EL \\ u_{\mathbf{t}}(x_{\mathbf{t}}^0) = 0,\ \mathbf{t} \in EL \\ \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) = 1.\end{array}\right\} (E(a,b,c,d))$$

The above constraints depend also on the alternatives $a,b,c,d \in A$ because their evaluations $g_{\mathbf{t}}(a)$, $g_{\mathbf{t}}(b)$, $g_{\mathbf{t}}(c)$ and $g_{\mathbf{t}}(d)$ give coordinates to four of $m_{\mathbf{t}}(A^R \cup \{a,b,c,d\})$ characteristic points of marginal value function $u_{\mathbf{t}}(\cdot)$, for each $\mathbf{t} \in EL$.

For all $a,b,c,d \in A$, and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, let us consider the following sets of constraints:

$$\left.\begin{array}{l} U(c) - U(d) \geq U(a) - U(b) + \varepsilon \\ E(a,b,c,d) \end{array}\right\} (E^N(a,b,c,d)), \qquad \left.\begin{array}{l} U(a) - U(b) \geq U(c) - U(d) \\ E(a,b,c,d) \end{array}\right\} (E^P(a,b,c,d)),$$

$$\left.\begin{array}{l} U_{\mathbf{r}}(c) - U_{\mathbf{r}}(d) \geq U_{\mathbf{r}}(a) - U_{\mathbf{r}}(b) + \varepsilon \\ E(a,b,c,d) \end{array}\right\} (E_{\mathbf{r}}^N(a,b,c,d)), \qquad \left.\begin{array}{l} U_{\mathbf{r}}(a) - U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(c) - U_{\mathbf{r}}(d) \\ E(a,b,c,d) \end{array}\right\} (E_{\mathbf{r}}^P(a,b,c,d)),$$

$$\left.\begin{array}{l} U_{\mathbf{t}}(c) - U_{\mathbf{t}}(d) \geq U_{\mathbf{t}}(a) - U_{\mathbf{t}}(b) + \varepsilon \\ E(a,b,c,d) \end{array}\right\} (E_{\mathbf{t}}^N(a,b,c,d)), \qquad \left.\begin{array}{l} U_{\mathbf{t}}(a) - U_{\mathbf{t}}(b) \geq U_{\mathbf{t}}(c) - U_{\mathbf{t}}(d) \\ E(a,b,c,d) \end{array}\right\} (E_{\mathbf{t}}^P(a,b,c,d)).$$

Thus, we get:

- $(a,b) \succsim^{*^N} (c,d) \Leftrightarrow E^N(a,b,c,d)$ is infeasible or $\varepsilon^N(a,b,c,d) \leq 0$, where $\varepsilon^N(a,b,c,d) = \max \varepsilon$, s.t.

constraints $E^N(a,b,c,d)$;

- $(a,b) \succsim^{*^P} (c,d) \Leftrightarrow$ if $E^P(a,b,c,d)$ is feasible and $\varepsilon^P(a,b,c,d) > 0$, where $\varepsilon^P(a,b,c,d) = \max \varepsilon$, s.t. constraints $E^P(a,b,c,d)$;
- $(a,b) \succsim^{*^N}_{\mathbf{r}} (c,d) \Leftrightarrow$ if $E^N_{\mathbf{r}}(a,b,c,d)$ is infeasible or $\varepsilon^N_{\mathbf{r}}(a,b,c,d) \leq 0$, where $\varepsilon^N_{\mathbf{r}}(a,b,c,d) = \max \varepsilon$, s.t. constraints $E^N_{\mathbf{r}}(a,b,c,d)$;
- $(a,b) \succsim^{*^P}_{\mathbf{r}} (c,d) \Leftrightarrow$ if $E^P_{\mathbf{r}}(a,b,c,d)$ is feasible and $\varepsilon^P_{\mathbf{r}}(a,b,c,d) > 0$, where $\varepsilon^P_{\mathbf{r}}(a,b,c,d) = \max \varepsilon$, s.t. constraints $E^P_{\mathbf{r}}(a,b,c,d)$;
- $(a,b) \succsim^{*^N}_{\mathbf{t}} (c,d) \Leftrightarrow$ if $E^N_{\mathbf{t}}(a,b,c,d)$ is infeasible or $\varepsilon^N_{\mathbf{t}}(a,b,c,d) \leq 0$, where $\varepsilon^N_{\mathbf{t}}(a,b,c,d) = \max \varepsilon$ s.t. constraints $E^N_{\mathbf{t}}(a,b,c,d)$;
- $(a,b) \succsim^{*^P}_{\mathbf{t}} (c,d) \Leftrightarrow$ if $E^P_{\mathbf{t}}(a,b,c,d)$ is feasible and $\varepsilon^P_{\mathbf{t}}(a,b,c,d) > 0$, where $\varepsilon^P_{\mathbf{t}}(a,b,c,d) = \max \varepsilon$, s.t. constraints $E^P_{\mathbf{t}}(a,b,c,d)$.

Most of the properties of quaternary relations $\succsim^{*^N}$ and $\succsim^{*^P}$, $\succsim^{*^N}_{\mathbf{r}}$ and $\succsim^{*^P}_{\mathbf{r}}$, and $\succsim^{*^N}_{\mathbf{t}}$ and $\succsim^{*^N}_{\mathbf{t}}$ are the same of those of the GRIP method presented in [8]. However, there are some properties specific to the case of the hierarchy of criteria, which are presented in the following proposition.

Proposition 5.1. *For all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$,

1. *given four alternatives* $a,b,c,d \in A$,

$$(a,b) \succsim^{*^N}_{(\mathbf{r},j)} (c,d), \forall j = 1,\ldots,n(\mathbf{r}) \Rightarrow (a,b) \succsim^{*^N}_{\mathbf{r}} (c,d);$$

2. *given four alternatives* $a,b,c,d \in A$ *such that:*
   (a) $(a,b) \succsim^{*^N}_{(\mathbf{r},j)} (c,d) \quad \forall j \in \{1,\ldots,n(\mathbf{r})\} \setminus \{w\}$,
   (b) $(a,b) \succsim^{*^P}_{(\mathbf{r},w)} (c,d)$,

   then $(a,b) \succsim^{*^P}_{\mathbf{r}} (c,d)$;

3. *given four alternatives* $a,b,c,d \in A$,

$$(a,b) \not\succsim^{*^P}_{(\mathbf{r},j)} (c,d) \;\; \forall j \in \{1,\ldots,n(\mathbf{r})\} \;\Rightarrow\; (a,b) \not\succsim^{*^P}_{\mathbf{r}} (c,d).$$

Proof. See Appendix. □

5.2 The representative value function

The ROR in case of the hierarchy of criteria builds a set of additive value functions compatible with preference information provided by the DM and leads to two preference relations, $\succsim^N_{\mathbf{r}}$ and $\succsim^P_{\mathbf{r}}$, for each subcriterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, from the hierarchy. Such preference relations answer to robustness concerns, since they are in general "more robust" than a preference relation determined by an arbitrarily chosen compatible value function. However, in practice, in some decision-making situations it is required to assign a score to considered alternatives. Moreover, possible and necessary preference relations may be not easy to interpret, even by a DM with some experience in MCDA. Thus, it is useful to determine a value function which represents well all the information contained in necessary and possible preference relations in an easily understandable way. For these reasons, a method for finding among all compatible value functions resulting from ROR a "representative" value function has been proposed in [10],[19]. It is based on the principle of "one for all, all for one", i.e. we look for one value function representing the set of all compatible value functions, and all compatible value functions contribute to define this representative value function.

In case of the hierarchy of criteria, the DM can be interested in a value function representing not only comprehensive necessary and possible preference relations, $\succsim^N$ and $\succsim^P$, but also necessary and possible preference relations $\succsim^N_{\mathbf{r}}$ and $\succsim^P_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, at intermediate levels. In general, the idea of the "representative value function" is to select from among compatible value functions that one which better highlights the necessary preference by maximizing the difference of values between alternatives $a, b \in A$ for which $a \succ^N b$, i.e. $a \succsim^N b$ and $b \not\succsim^N a$. As secondary objective, one can consider minimizing the difference of values between alternatives $a, b \in A$ for which $a \not\succsim^N b$ and $b \not\succsim^N a$. In case of the hierarchy of criteria one can imagine that the DM gives a sequence of criteria $G_{\mathbf{r}_1}, \ldots, G_{\mathbf{r}_f} \in \mathcal{G}$, ordered with respect to his/her interest. In this case, the representative value function is the one maximizing the difference of values between alternatives $a, b \in A$ for which $a \succ^N_{\mathbf{r}_i} b$, and minimizing the difference of values between alternatives $a, b \in A$ for which $a \not\succsim^N_{\mathbf{r}_i} b$ and $b \not\succsim^N_{\mathbf{r}_i} a$, starting from the most interesting subcriterion $G_{\mathbf{r}_1}$ and proceeding in the above sequence until subcriterion $G_{\mathbf{r}_f}$. In this way, the discrimination power of the "representative value function" is maximal for the most interesting subcriterion $G_{\mathbf{r}_1}$, and it is decreasing, step by step, until subcriterion $G_{\mathbf{r}_f}$. Summing up, the "representative" value function can be found via the following procedure:

1. Consider the set of constraints E^A including constraints representing preference information provided by the DM, and monotonicity constraints on marginal value functions $u_{\mathbf{t}}(\cdot)$, $\mathbf{t} \in EL$, whose characteristic points correspond to all different evaluations of alternatives from set A (and not only from the

reference subset $A^R \subseteq A$) on particular elementary criteria:

$$\left.\begin{array}{l}\left.\begin{array}{l}U(a^*) \geq U(b^*) + \varepsilon \quad \text{if} \;\; a^* \succ b^* \\ U(a^*) = U(b^*) \quad \text{if} \;\; a^* \sim b^* \\ U(a^*) - U(b^*) \geq U(c^*) - U(d^*) + \varepsilon \quad \text{if} \;\; (a^*, b^*) \succ^* (c^*, d^*) \\ U(a^*) - U(b^*) = U(c^*) - U(d^*) \quad \text{if} \;\; (a^*, b^*) \sim^* (c^*, d^*) \\ U_{\mathbf{r}}(a^*) \geq U_{\mathbf{r}}(b^*) + \varepsilon \quad \text{if} \;\; a^* \succ_{\mathbf{r}} b^* \\ U_{\mathbf{r}}(a^*) = U_{\mathbf{r}}(b^*) \quad \text{if} \;\; a^* \sim_{\mathbf{r}} b^* \\ U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) \geq U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) + \varepsilon \quad \text{if} \;\; (a^*, b^*) \succ^*_{\mathbf{r}} (c^*, d^*) \\ U_{\mathbf{r}}(a^*) - U_{\mathbf{r}}(b^*) = U_{\mathbf{r}}(c^*) - U_{\mathbf{r}}(d^*) \quad \text{if} \;\; (a^*, b^*) \sim^*_{\mathbf{r}} (c^*, d^*) \end{array}\right\} a^*, b^*, c^*, d^* \in A^R, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL, \\ u_{\mathbf{t}}(x^k_{\mathbf{t}}) - u_{\mathbf{t}}(x^{k-1}_{\mathbf{t}}) \geq 0, \;\; \forall \mathbf{t} \in EL, \;\; k = 1, ..., m_{\mathbf{t}} \\ u_{\mathbf{t}}(x^1_{\mathbf{t}}) = 0, \;\; \forall \mathbf{t} \in EL \\ \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}) = 1, \end{array}\right\} (E^A)$$

where, $x^1_{\mathbf{t}} = min_{a \in A} \; g_{\mathbf{t}}(a)$, and $x^{m_{\mathbf{t}}}_{\mathbf{t}} = max_{a \in A} \; g_{\mathbf{t}}(a)$; $x^k_{\mathbf{t}} \in X_{\mathbf{t}}, k = 1, ..., m_{\mathbf{t}}$, with $X_{\mathbf{t}}$ the set of all different evaluations of alternatives from A on elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in EL$, and $m_{\mathbf{t}} = |X_{\mathbf{t}}|$. The values $x^k_{\mathbf{t}}, k = 1, ..., m_{\mathbf{t}}$, are increasingly ordered, i.e.,

$$x^1_{\mathbf{t}} < x^2_{\mathbf{t}} < ... < x^{m_{\mathbf{t}}-1}_{\mathbf{t}} < x^{m_{\mathbf{t}}}_{\mathbf{t}}.$$

2. Calculate $\varepsilon^* = \max \;\; \varepsilon, \;\; \text{s.t.} \;\; E^A$. If $\varepsilon^* > 0$, then there exists at least one value function satisfying constraints of E^A, so go to step 3. If $\varepsilon^* \leq 0$, then there is no value function satisfying E^A, which means that the information provided by the DM cannot be faithfully represented by any additive value function. If the DM accepts to work with not fully compatible value functions, then go to step 3; if the DM decides to remove a part of preference information causing the incompatibility, then after this removal (see section 7), go to step 3,

3. $i = 1$; $E = E^A$,

4. Determine the necessary preference relation $\succsim^N_{\mathbf{r}_i}$ and the possible preference relation $\succsim^P_{\mathbf{r}_i}$ with respect to subcriterion $G_{\mathbf{r}_i} \in \mathcal{G}$, considering the sets of constraints:

$$\left.\begin{array}{l} U_{\mathbf{r}_i}(b) \geq U_{\mathbf{r}_i}(a) + \varepsilon \\ E^A \end{array}\right\} (E^N_{\mathbf{r}_i}(a, b)), \qquad \left.\begin{array}{l} U_{\mathbf{r}_i}(a) \geq U_{\mathbf{r}_i}(b) \\ E^A \end{array}\right\} (E^P_{\mathbf{r}_i}(a, b)).$$

- $a \succsim_{\mathbf{r}_i}^N b \Leftrightarrow$ the set $E_{\mathbf{r}_i}^N(a,b)$ is infeasible or $\varepsilon_{\mathbf{r}_i}^{*,N} \leq 0$, where $\varepsilon_{\mathbf{r}_i}^{*,N} = \max \varepsilon$, s.t. constraints $E_{\mathbf{r}_i}^N(a,b)$,
- $a \succsim_{\mathbf{r}_i}^P b \Leftrightarrow E_{\mathbf{r}_i}^P(a,b)$ is feasible and $\varepsilon_{\mathbf{r}_i}^{*,P} > 0$, where $\varepsilon_{\mathbf{r}_i}^{*,P} = \max \varepsilon$, s.t. constraints $E_{\mathbf{r}_i}^P(a,b)$.

5. For all pairs of alternatives (a,b), such that $a \succ_{\mathbf{r}_i}^N b$, add the following constraint to E: $U_{\mathbf{r}_i}(a) \geq U_{\mathbf{r}_i}(b) + \varepsilon_{\mathbf{r}_i}$, i.e.,

$$\left.\begin{array}{l} E \\ U_{\mathbf{r}_i}(a) \geq U_{\mathbf{r}_i}(b) + \varepsilon_{\mathbf{r}_i} \quad \text{if } \; a \succ_{\mathbf{r}_i} b \end{array}\right\} \rightarrow (E)$$

   if $i=1$, then go to step 6, otherwise go to step 7,

6. Add constraint $\varepsilon_{\mathbf{r}_i} = \varepsilon$ to E,

$$\left.\begin{array}{l} E \\ \varepsilon_{\mathbf{r}_i} = \varepsilon \end{array}\right\} \rightarrow (E)$$

7. Maximize $\varepsilon_{\mathbf{r}_i}$, subject to constraints E.

8. Add the constraint $\varepsilon_{\mathbf{r}_i} = \varepsilon_{\mathbf{r}_i}^*$ to E, with $\varepsilon_{\mathbf{r}_i}^* = \max \varepsilon_{\mathbf{r}_i}$ computed in step 7,

$$\left.\begin{array}{l} E \\ \varepsilon_{\mathbf{r}_i} = \varepsilon_{\mathbf{r}_i}^* \end{array}\right\} \rightarrow (E)$$

9. For all pairs of alternatives (a,b), such that $a \not\succsim_{\mathbf{r}_i}^N b$ and $b \not\succsim_{\mathbf{r}_i}^N a$ (already computed in step 4), add the following constraints to E: $U_{\mathbf{r}_i}(a) - U_{\mathbf{r}_i}(b) \leq \delta_{\mathbf{r}_i}$ and $U_{\mathbf{r}_i}(b) - U_{\mathbf{r}_i}(a) \leq \delta_{\mathbf{r}_i}$,

$$\left.\begin{array}{l} E \\ \left.\begin{array}{l} U_{\mathbf{r}_i}(a) - U_{\mathbf{r}_i}(b) \leq \delta_{\mathbf{r}_i} \\ U_{\mathbf{r}_i}(b) - U_{\mathbf{r}_i}(a) \leq \delta_{\mathbf{r}_i} \end{array}\right\} \; \text{if } \; a \not\succsim_{\mathbf{r}_i}^N b \; \text{ and } \; b \not\succsim_{\mathbf{r}_i}^N a \end{array}\right\} \rightarrow (E)$$

10. Minimize $\delta_{\mathbf{r}_i}$, subject to constraints E.

11. Add the constraint $\delta_{\mathbf{r}_i} = \delta_{\mathbf{r}_i}^*$ to E, with $\delta_{\mathbf{r}_i}^* = \min \delta_{\mathbf{r}_i}$ computed in step 10,

$$\left.\begin{array}{l} E \\ \delta_{\mathbf{r}_i} = \delta_{\mathbf{r}_i}^* \end{array}\right\} \rightarrow (E)$$

12. If $i < f$ then go to step 4 with $i := i+1$, otherwise stop.

Observe that the above procedure takes into account the preference information given by the DM by maximizing the value of auxiliary variable ε in the first iteration. This ensures that the DM's preferences are represented with a maximal discrimination possible. If the DM does not want to express a sequence of subcriteria $G_{\mathbf{r}_1},\ldots,G_{\mathbf{r}_f}\in\mathcal{G}$, but (s)he wants to compute the representative value function considering only the comprehensively necessary preference relation, it will be enough to perform a single iteration of the procedure described until step 10, considering $i=1$ and $\mathbf{r}_1=0$.

Let us mention that other methods proposed for finding a representative value function in ordinal regression [2, 3], not referring to necessary and possible preference relations, can also be adapted to the case of hierarchy of criteria.

6 A didactic example

In this section, we apply the procedure described in the previous sections to cope with a hierarchical multiple criteria decision problem which is very frequent in the scholar system, and in the academic sector in particular. Let us suppose that each year a faculty of natural sciences has the economic possibility to give a scholarship to one of its best students; to make the choice, the Dean is considering fifteen students who attended the courses and passed the test of two macro subjects: Mathematics and Chemistry. Mathematics has two sub-subjects: Algebra and Analysis, while Chemistry has two sub-subjects: Analytical Chemistry and Organic Chemistry; each of these sub-subjects has other two sub-subjects for a total of eight elementary sub-subjects shown in Figure 2. Using the terminology introduced in Section 2, the set of alternatives $A=\{\mathbf{A},\mathbf{B},\ldots,\mathbf{R}\}$ is composed of 15 alternatives; the number of levels $l=3$; the set of all criteria $\mathcal{G}=\{G_1,G_2,\ G_{(1,1)},G_{(1,2)},\ G_{(2,1)},G_{(2,2)},\ g_{(1,1,1)},g_{(1,1,2)},\ g_{(1,2,1)},g_{(1,2,2)},\ g_{(2,1,1)},g_{(2,1,2)},\ g_{(2,2,1)},g_{(2,2,2)}\}$ is composed of criteria and subcriteria whose names are given in Figure 2; the set of indices of all criteria is $\mathcal{I}_{\mathcal{G}}=\{1,2,\ (1,1),(1,2),\ (2,1),(2,2),\ (1,1,1),(1,1,2),\ (1,2,1),(1,2,2),\ (2,1,1),(2,1,2),\ (2,2,1),(2,2,2)\}$; the number of first level criteria $m=2$; if we consider $G_{\mathbf{r}}=G_1$ then $n(\mathbf{r})=2$, while if we consider $G_{\mathbf{r}}=G_{(1,1)}$ then $n((1,1))=2$; $g_{(1,1,1)},g_{(1,1,2)},g_{(1,2,1)},g_{(1,2,2)},g_{(2,1,1)},g_{(2,1,2)},g_{(2,2,1)},g_{(2,2,2)}$ are the elementary subcriteria; the set of indices of elementary subcriteria is $EL=\{(1,1,1),(1,1,2),\ (1,2,1),(1,2,2),\ (2,1,1),(2,1,2),\ (2,2,1),(2,2,2)\}$; if we consider $G_{\mathbf{r}}=G_1$ then $E(G_{(1)})=\{(1,1,1),(1,1,2),\ (1,2,1),(1,2,2)\}$ while if we consider $G_{\mathbf{r}}=G_{(2,1)}$ then $E(G_{(2,1)})=\{(2,1,1),(2,1,2)\}$.

As it was declared in Section 2, the students are evaluated directly on the elementary subcriteria only, and thus, they are evaluated with respect to the eight elementary sub-subjects; these evaluations are shown in Table 1. Each elementary subcriterion has five qualitative levels of evaluation that go from very bad to very good, increasingly ordered.

The only comprehensive relation that comes out from the problem formulation is the dominance relation in the set of students, shown in Figure 3. The dominance relation does not take into account the preferences

Figure 2: Hierarchical structure of criteria

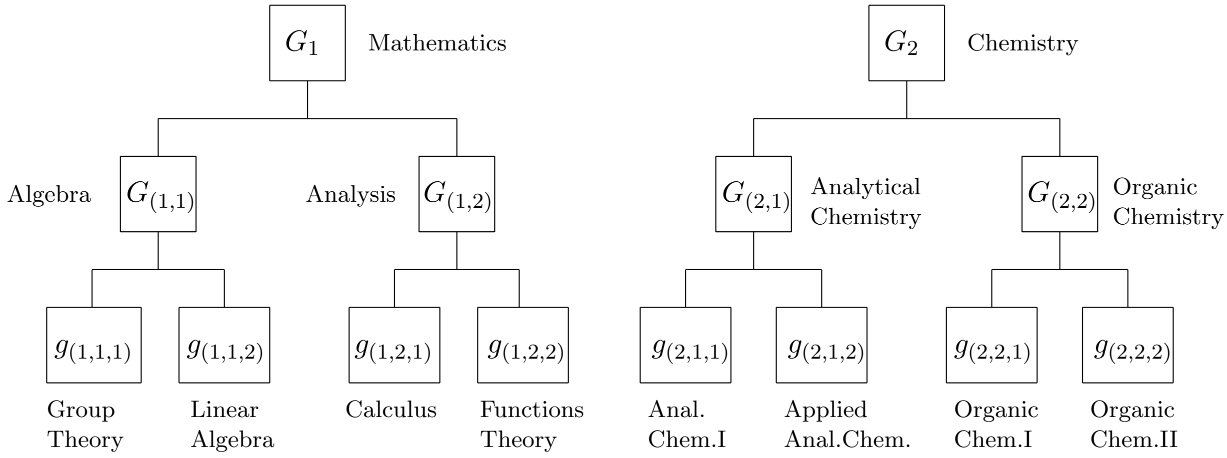
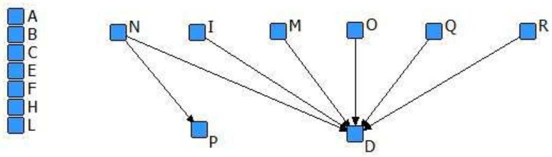


Table 1: Evaluations of students on the eight elementary subcriteria

student\subcriteria	$g_{(1,1,1)}$	$g_{(1,1,2)}$	$g_{(1,2,1)}$	$g_{(1,2,2)}$	$g_{(2,1,1)}$	$g_{(2,1,2)}$	$g_{(2,2,1)}$	$g_{(2,2,2)}$
A	Very Bad	Very Good	Very Bad	Good	Very Good	Very Good	Very Bad	Bad
B	Bad	Very Good	Medium	Very Good	Very Bad	Bad	Very Bad	Very Bad
C	Very Good	Medium	Medium	Very Bad	Very Good	Good	Bad	Medium
D	Medium	Very Bad	Bad	Very Bad	Very Bad	Bad	Medium	Very Bad
E	Very Good	Very Good	Medium	Medium	Bad	Very Good	Bad	Very Bad
F	Good	Bad	Bad	Medium	Very Bad	Very Bad	Very Good	Very Good
H	Medium	Very Bad	Bad	Bad	Very Good	Very Bad	Very Bad	Very Bad
I	Good	Good	Good	Medium	Medium	Bad	Good	Very Bad
L	Good	Very Bad	Bad	Good	Good	Very Bad	Very Good	Good
M	Medium	Medium	Medium	Bad	Medium	Medium	Very Good	Good
N	Good	Bad	Very Good	Medium	Bad	Very Good	Very Good	Medium
O	Good	Medium	Bad	Bad	Medium	Bad	Very Good	Very Bad
P	Bad	Very Bad	Bad	Medium	Bad	Very Good	Medium	Very Bad
Q	Very Good	Very Good	Medium	Very Bad	Bad	Medium	Medium	Bad
R	Good	Good	Bad	Very Bad	Bad	Bad	Medium	Medium

Figure 3: Dominance relation in the set of students



of the Dean and, moreover, it leaves too many students incomparable. For this reason, the Dean decides to use the ROR approach adapted to the hierarchical structure of criteria.

The Dean provides the following preference information which is then transformed to constraints of the ordinal regression problem:

1. On Chemistry, student **I** is preferred to student **H**. In order to take into consideration this preference

Figure 4: Necessary preference relation determined by the first piece of preference information

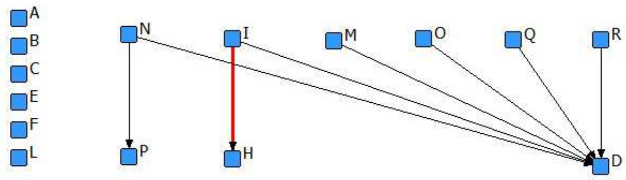
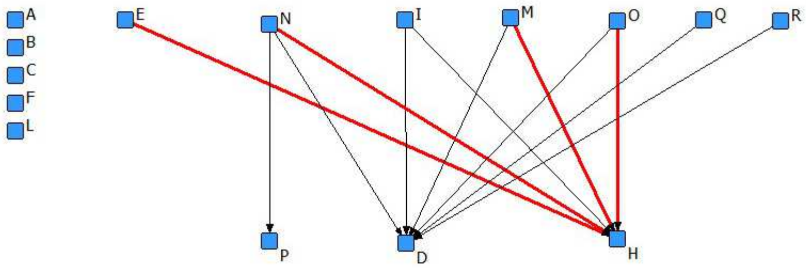


Figure 5: Necessary preference relation determined by the two pieces of preference information



information it is represented in the constraints (E^{A^R}) as follows:

$$U_2(I) > U_2(H) \Leftrightarrow U_{(2,1)}(\mathbf{I}) + U_{(2,2)}(\mathbf{I}) > U_{(2,1)}(\mathbf{H}) + U_{(2,2)}(\mathbf{H}) \quad \Leftrightarrow$$

$$\Leftrightarrow u_{(2,1,1)}(\mathbf{I}) + u_{(2,1,2)}(\mathbf{I}) + u_{(2,2,1)}(\mathbf{I}) + u_{(2,2,2)}(\mathbf{I}) > u_{(2,1,1)}(\mathbf{H}) + u_{(2,1,2)}(\mathbf{H}) + u_{(2,2,1)}(\mathbf{H}) + u_{(2,2,2)}(\mathbf{H}).$$

Figure 4 shows the necessary preference relation determined by this piece of preference information. In Figure 4, the arrow from **I** to **H** is bold marked because it constitutes the part of necessary preference relation originating from the considered piece of preference information and, therefore, not present at the previous stage (dominance relation, see Figure 3). Bold marked arrows in the following figures have an analogous interpretation with respect to preference information provided in further steps.

2. On Analytical Chemistry, student **E** is preferred to student **H**. This, can be modeled using the following constraint:

$$U_{(2,1)}(\mathbf{E}) > U_{(2,1)}(\mathbf{H}) \Leftrightarrow u_{(2,1,1)}(\mathbf{E}) + u_{(2,1,2)}(\mathbf{E}) > u_{(2,1,1)}(\mathbf{H}) + u_{(2,1,2)}(\mathbf{H}).$$

Figure 5 shows the necessary preference relation determined by the two pieces of preference information.

3. On Mathematics, student **N** is preferred to student **Q**. This, can be modeled using the following constraint:

$$U_1(\mathbf{N}) > U_1(\mathbf{Q}) \Leftrightarrow U_{(1,1)}(\mathbf{N}) + U_{(1,2)}(\mathbf{N}) > U_{(1,1)}(\mathbf{Q}) + U_{(1,2)}(\mathbf{Q}) \quad \Leftrightarrow$$

Figure 6: Necessary preference relation determined by the three pieces of preference information

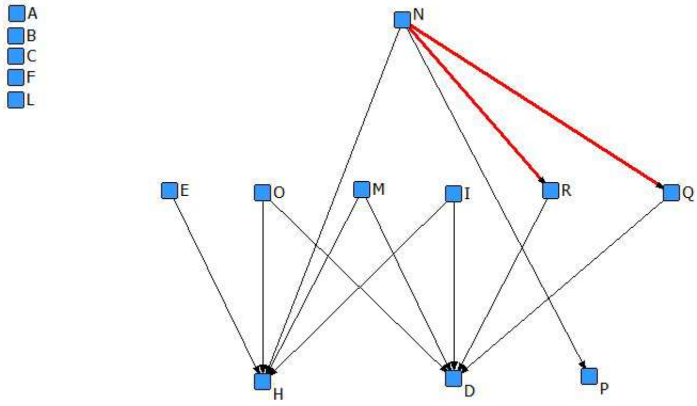
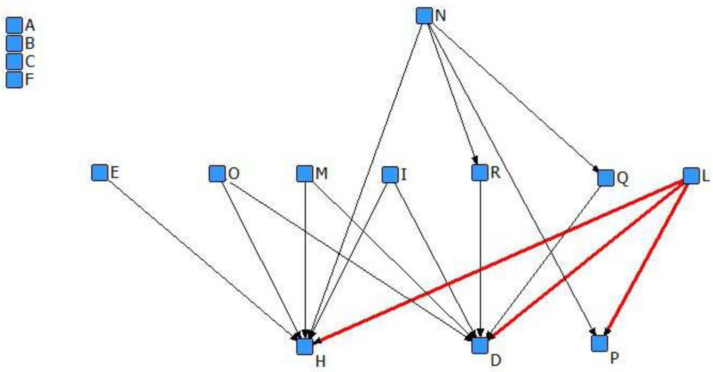


Figure 7: Necessary preference relation determined by the four pieces of preference information



$$\Leftrightarrow u_{(1,1,1)}(\mathbf{N})+u_{(1,1,2)}(\mathbf{N})+u_{(1,2,1)}(\mathbf{N})+u_{(1,2,2)}(\mathbf{N}) > u_{(1,1,1)}(\mathbf{Q})+u_{(1,1,2)}(\mathbf{Q})+u_{(1,2,1)}(\mathbf{Q})+u_{(1,2,2)}(\mathbf{Q}).$$

Figure 6 shows the necessary preference relation determined by the three pieces of preference information.

4. On Chemistry, student **L** is preferred to student **P**. This, can be modeled using the following constraint:

$$U_2(\mathbf{L}) > U_2(\mathbf{P}) \Leftrightarrow U_{(2,1)}(\mathbf{L}) + U_{(2,2)}(\mathbf{L}) > U_{(2,1)}(\mathbf{P}) + U_{(2,2)}(\mathbf{P}) \quad \Leftrightarrow$$

$$\Leftrightarrow u_{(2,1,1)}(\mathbf{L})+u_{(2,1,2)}(\mathbf{L})+u_{(2,2,1)}(\mathbf{L})+u_{(2,2,2)}(\mathbf{L}) > u_{(2,1,1)}(\mathbf{P})+u_{(2,1,2)}(\mathbf{P})+u_{(2,2,1)}(\mathbf{P})+u_{(2,2,2)}(\mathbf{P}).$$

Figure 7 shows the necessary preference relation determined by the four pieces of preference information.

In the context of the hierarchical multiple criteria evaluation, it is possible to check the necessary preference relation at intermediate levels of the hierarchy, that is we can see if student a is necessarily preferred to student b with respect to considered domain (Mathematics, Chemistry, Algebra, Analysis and so on); in Tables 2 and 3, we present the necessary preference relation with respect to macro subjects: Mathematics and Chemistry, respectively.

Table 2: Necessary preference relations for Mathematics and its subcriteria

student\subcriterion	$\succsim_{\mathbf{1}}^{\mathbf{N}}$	$\succsim_{(\mathbf{1},\mathbf{1})}^{\mathbf{N}}$	$\succsim_{(\mathbf{1},\mathbf{2})}^{\mathbf{N}}$
A			
B	**A, P**	**A, P**	**A, C, D, E, F, H, L, M, O, P, Q, R**
C	**D**	**D, F, H, L, M, N, O, P**	**D, Q, R**
D		**H, P**	**R**
E	**C, D, F, H, M, O, P, Q, R**	**A, B, C, D, F, H, I, L, M, N, O, P, Q, R**	**C, D, F, H, M, O, P, Q, R**
F	**D, H, P**	**D, H, L, N, P**	**D, H, O, P, R**
H	**D**	**D, P**	**D, O, R**
I	**D, F, H, M, O, P, R**	**D, F, H, L, M, N, O, P, R**	**C, D, E, F, H, M, O, P, Q, R**
L	**D, H, P**	**D, H, P**	**A, D, F, H, O, P, R**
M	**D, H**	**D, H, P**	**C, D, H, O, Q, R**
N	*C*,**D, F, H, P**,*Q*,*R*	**D, F, H, L, P**	**C, D, E, F, H, I, M, O, P, Q, R**
O	**D, H**	**D, F, H, L, M, N, P**	**D, H, R**
P			**D, F, H, O, R**
Q	**C, D, R**	**A, B, C, D, E, F, H, I, L, M, N, O, P, R**	**C, D, R**
R	**D**	**D, F, H, I, L, M, N, O, P**	**D**

Table 3: Necessary preference relation for Chemistry and its subcriteria

student\subcriterion	$\succsim_{\mathbf{2}}^{\mathbf{N}}$	$\succsim_{(\mathbf{2},\mathbf{1})}^{\mathbf{N}}$	$\succsim_{(\mathbf{2},\mathbf{2})}^{\mathbf{N}}$
A	**B, H**	**B, C, D, E, F, H, I, L, M, N, O, P, Q, R**	**B, H**
B		**D, F**	**H**
C	**B, H**	**B, D, F, H, I, L, M, O, Q, R**	**A, B, E, H**
D	**B**	**B, F**	**B, E, H, P**
E	**B, H**	**B, D, F, H, L, N, P, Q, R**	**B, H**
F			**A, B, C, D, E, H, I, L, M, N, O, P, Q, R**
H		**F, L**	**B**
I	**B, D**,*H*	**B, D, F, O, R**	**B, D, E, H, P**
L	*B,D,H,P*	**F**	**A, B, C, D, E, H, I, M, N, O, P, Q, R**
M	**B, D, H, I, O, Q, R**	**B, D, F, I, O, Q, R**	**A, B, C, D, E, H, I, L, N, O, P, Q, R**
N	**B, D, E, H, P, Q, R**	**B, D, E, F, H, L, P, Q, R**	**A, B, C, D, E, H, I, O, P, Q, R**
O	**B, D**,*H*,**I**	**B, D, F, I, R**	**B, D, E, H, I, P**
P	**B, D, E, H**	**B, D, E, F, H, L, N, Q, R**	**B, D, E, H**
Q	**B, D**	**B, D, F, R**	**A, B, D, E, H, P**
R	**B, D**	**B, D, F**	**A, B, C, D, E, H, P, Q**

In Tables 2 and 3, the alternatives in italics are those for which the necessary preference relation is true at the second level but it is not true at the level below. For example, $\mathbf{L} \succsim_2^N \mathbf{B}$ but $\mathbf{L} \not\succsim_{(2,1)}^N \mathbf{B}$.

As shown in subsection 5.2, one can compute the representative value function, taking into account a sequence of subcriteria $G_{\mathbf{r}_1}, \ldots, G_{\mathbf{r}_f} \in \mathcal{G}$ ordered with respect to the Dean's interest. Results presented in Table 4 show the ranking of students obtained using the representative value function in three different cases:

- the Dean considers as the most important and the second most important the criteria Mathematics (G_1) and Chemistry (G_2), respectively, and consequently, he considers the sequence of corresponding necessary preference relations $\succsim_1^N, \succsim_2^N$ (1^{st} and 2^{nd} columns),

- the Dean considers as the most important and the second most important the criteria Chemistry (G_2) and Mathematics (G_1), respectively, and consequently, he considers the sequence of corresponding necessary preference relations $\succsim_2^N, \succsim_1^N$ (3^{rd} and 4^{th} columns),

- the Dean does not discriminate criteria with respect to their importance and consequently he takes into account only the comprehensive necessary preference relation $\succsim_0^N$ (5^{th} column).

We can observe three important facts:

- student **N** is almost always the best one in the ranking obtained using different representative value functions,
- the ranking obtained by the representative value function changes between the first and the second iteration of the method,
- the ranking obtained by the representative value function changes if we consider a different order of importance between the necessary preference relations.

Table 4: Ranking of students by a representative value function (in parentheses there are value of the corresponding alternatives)

$\succsim_{\mathbf{r_1}}^N=\succsim_1^N$	$\succsim_{\mathbf{r_2}}^N=\succsim_2^N$		$\succsim_{\mathbf{r_1}}^N=\succsim_2^N$	$\succsim_{\mathbf{r_2}}^N=\succsim_1^N$		$\succsim^N$
N(0.8560)	**N**(0.8586)		**N**(1)	**N**(1)		**M**(0.8808)
I(0.6635)	**I**(0.6949)		**M**(0.8752)	**M**(0.8636)		**N**(0.8622)
E(0.6250)	**E**(0.6250)		**L**(0.7663)	**L**(0.7273)		**F**(0.6690)
M(0.6023)	**M**(0.5881)		**O**(0.6934)	**O**(0.6818)		**L**(0.6690)
Q(0.5611)	**Q**(0.5453)		**F**(0.6754)	**F**(0.6364)		**A**(0.6690)
F(0.5)	**F**(0.5)		**P**(0.5844)	**P**(0.5455)		**I**(0.5426)
L(0.5)	**L**(0.5)		**I**(0.5735)	**I**(0.5)		**C**(0.4915)
C(0.4773)	**C**(0.4590)		**Q**(0.4940)	**Q**(0.4944)		**O**(0.4893)
B(0.4630)	**A**(0.4572)		**A**(0.4875)	**A**(0.4489)		**R**(0.4654)
A(0.4588)	**O**(0.4474)		**R**(0.4091)	**R**(0.4091)		**Q**(0.4617)
O(0.4559)	**B**(0.4389)		**E**(0.4026)	**E**(0.3636)		**P**(0.4190)
R(0.4087)	**R**(0.3678)		**C**(0.3621)	**C**(0.3567)		**E**(0.4190)
P(0.25)	**P**(0.25)		**D**(0.2273)	**D**(0.2273)		**B**(0.3808)
H(0.1250)	**H**(0.125)		**H**(0.1934)	**H**(0.1818)		**D**(0.2117)
D(0.0880)	**D**(0.0639)		**B**(0.1754)	**B**(0.1364)		**H**(0.1690)

7 Further extensions of ROR for the hierarchy of criteria

Infeasibility

We have seen in section 3, that the first step of ROR is to check if there exists at least one value function compatible with the preference information provided by the DM. In fact, it is possible that the information provided by the DM is such that it is not possible to find a compatible additive value function. In this case, the DM, together with the analyst, can decide to continue the study while accepting to work with not fully compatible value functions, or look for sets of constraints responsible of this infeasibility (let us call them troublesome constraints), and remove them from the linear program.

In case of the hierarchy of criteria, inconsistencies can be present at different levels of the hierarchy and for this reason, differently from [21] where all constraints translate preference information concerning the same level, the DM could be interested in removing troublesome constraints regarding a particular set of criteria/subcriteria $\{G_{\mathbf{r_1}}, \ldots, G_{\mathbf{r_h}}\}$. For example, considering preference information regarding students evaluated on criteria structured according to the hierarchy shown in Section 6, the DM could be interested in removing the troublesome constraints at the lowest level possible, i.e. starting by the last but one level, that is constraints regarding Algebra, Analysis, Analytical Chemistry and Organic Chemistry. Then, if it is still not sufficient to get feasibility of the whole set of constraints E^{A^R}, one can look at the constraints of the level immediately above, that is constraints regarding Mathematics and Chemistry, and so on; in this way (s)he could examine the infeasibility going up the hierarchy of criteria. Another DM could be interested in removing troublesome constraints regarding sets of criteria from different levels, like, for example, {Mathematics, Organic Chemistry} or {Analysis, Analytical Chemistry}, or Mathematics alone, or Chemistry alone, and so on. Two important remarks concerning this procedure have to be done:

- looking for troublesome constraints among all constraints E^{A^R} translating the full preference information provided by the DM can be seen as a particular case of the above procedure; in fact, in order to get the whole set of constraints E^{A^R}, it is sufficient to consider the set $\{G_{\mathbf{r_1}}, \ldots, G_{\mathbf{r_h}}\}$ of criteria composed of all criteria from the first level of the hierarchy,

- finding a set of troublesome constraints regarding a particular set of criteria/subcriteria could be not sufficient to remove the infeasibility of the whole set of constraints E^{A^R}; if it would be the case, one should continue the search and add some criterion/subcriterion to the set of criteria $\{G_{\mathbf{r_1}}, \ldots, G_{\mathbf{r_h}}\}$ considered before in order to verify if removing troublesome constraints from the extended set is sufficient to make E^{A^R} feasible.

Knowing a few or all sets of constraints causing infeasibility, if the DM would refuse to choose the one to be removed, then the analyst could suggest a certain heuristic for ordering these sets of constraints with respect to importance of the corresponding piece of preference information. For example, given a set of constraints $S = \{C_1, \ldots, C_p\}$ coming from levels $\{h_1, \ldots, h_p\}$, respectively, one could associate to this set the number $H_S = \left(\sum_{k=1}^{p} h_k\right)/p$. H_S represents an average level of constraints belonging to set S. Supposing that a constraint from level h is more important than the one from level $h+1$, one could decide to remove S_i, such that $H_{S_i} > H_{S_j}$ for all $j \neq i$, that is the set having the greatest value of H_{S_i}. If two sets, S_i and S_j, would have the same score $H_{S_i} = H_{S_j}$, then we could remove the one that has less constraints coming from the lowest level. In order to find sets of troublesome constraints in a set of constraints translating preference information, one can proceed as shown in [21].

Credibility

ROR methods permit to specify incrementally the preferences of the DM, assigning them a different degree of credibility. The idea of considering a sequence of pieces of preference information ordered according to their credibility has been introduced in [12] and investigated further in [18]. More formally, the preference information given by the DM is represented as a chain of embedded preference relations $\succsim_1 \subseteq \ldots \subseteq \succsim_n$, where for each $r, s = 1, \ldots, n$, with $r < s$, the preference $\succsim_r$ is more credible than $\succsim_s$. If for any $t = 1, \ldots, n$, we denote by E_t the set of constraints obtained from $\succsim_t$, and by $\mathcal{U}_t$ the sets of value functions compatible with the preference information of $\succsim_t$, then we have $E_1 \subseteq \ldots \subseteq E_n$ and $\mathcal{U}_1 \supseteq \ldots \supseteq \mathcal{U}_n$, and consequently $\succsim_1^N \subseteq \ldots \subseteq \succsim_n^N$, and $\succsim_1^P \supseteq \ldots \supseteq \succsim_n^P$, that is the smaller the credibility of the considered preference relation $\succsim_t$, the richer the necessary preference relation $\succsim_t^N$ and the poorer the possible preference relation $\succsim_t^P$. In case of the hierarchy of criteria, for each subcriterion $G_{\mathbf{r}} \in \mathcal{G}$, we have a sequence of nested possible preference relations $\succsim_{\mathbf{r},1}^P \supseteq \ldots \supseteq \succsim_{\mathbf{r},n}^P$ and a sequence of nested necessary preference relations $\succsim_{\mathbf{r},1}^N \subseteq \ldots \subseteq \succsim_{\mathbf{r},n}^N$.

Extreme ranking

Necessary and possible preference relations give information regarding couples of alternatives. However, it could be interesting to analyse some information related to the whole set of alternatives in terms of the best and the worst ranking position assigned to each alternative by the compatible value functions. This constitutes the extreme ranking analysis introduced in [18]. In case of the hierarchy of criteria, the extreme ranking analysis can be performed for each subcriterion $G_{\mathbf{r}} \in \mathcal{G}$.

UTADIS^{GMS}

In general, MCDA considers three types of problems:

- ranking, consisting in completely or partially ordering the alternatives from the best to the worst,
- choice, consisting in selecting a subset of the best alternatives,
- sorting, consisting in assigning the alternatives to some predefined and preferentially ordered classes.

Ranking and choice problems are based on pairwise comparisons of alternatives and, therefore, they can be dealt with possible and necessary preference relations. Sorting relies instead on the intrinsic value of an alternative and not on its comparison to others. Therefore, sorting problems need specific methods. Within ROR, UTADIS^{GMS} [14] has been proposed to deal with sorting problems as follows. Given a set of pre-defined classes $C_1, C_2, \ldots, C_p$, ordered from the worst to the best, the DM gives preference information in terms of exemplary assignments of reference alternatives to some sequences of classes, such that $a^* \rightarrow [C_{L^{DM}}(a^*), C_{R^{DM}}(a^*)]$, with $L^{DM} \leq R^{DM}$, means that reference alternative a^* can be assigned to one of the classes between $C_{L^{DM}}(a^*)$ and $C_{R^{DM}}(a^*)$. Denoting by $A^R \subseteq A$ the set of reference alternatives considered by the DM, we say that a value function U is compatible if

$$\forall a^*, b^* \in A^R, L^{DM}(a^*) > R^{DM}(b^*) \Rightarrow U(a^*) > U(b^*). \tag{3}$$

Denoting by $\mathcal{U}$ the set of compatible value functions, we have that each $U \in \mathcal{U}$ assigns an alternative $a \in A$ to a sequence of classes $\left[L^U(a), R^U(a)\right]$, where

$$L^U(a) = \max\left(\{1\} \cup \left\{L^{DM}(a^*) : U(a^*) \leq U(a), a^* \in A^R\right\}\right),$$

$$R^U(a) = \min\left(\{p\} \cup \left\{R^{DM}(a^*) : U(a^*) \geq U(a), a^* \in A^R\right\}\right).$$

Within ROR, considering the set of all compatible value functions, for each $a \in A$ one can define the possible assignment $C^P(a)$ and the necessary assignment $C^N(a)$ as follows:

- $C^P(a) = \left[L_P^{\mathcal{U}}(a), R_P^{\mathcal{U}}(a)\right] = \cup_{U \in \mathcal{U}} \left[L^U(a), R^U(a)\right]$,
- $C^N(a) = \left[L_N^{\mathcal{U}}(a), R_N^{\mathcal{U}}(a)\right] = \cap_{U \in \mathcal{U}} \left[L^U(a), R^U(a)\right]$.

In case of the hierarchy of criteria, the DM can give exemplary assignments $a^* \rightarrow [C_{L^{DM}}(a^*), C_{R^{DM}}(a^*)]$ at a comprehensive level, but (s)he can also give assignments $a^* \rightarrow_{\mathbf{r}} \left[C^{\mathbf{r}}_{L_{\mathbf{r}}^{DM}}(a^*), C^{\mathbf{r}}_{R_{\mathbf{r}}^{DM}}(a^*)\right]$ with respect to each subcriterion $G_{\mathbf{r}}$ from the hierarchy, excluding the elementary subcriteria, i.e. $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$.

For example, suppose that a Dean has to evaluate students according to their scores in various subjects. He can say that student s_1 is assigned comprehensively to a class between "Medium" and "Very good", i.e. $s_1 \rightarrow$ [Medium,Very good], but he can also say that student s_2 (not necessarily s_2 different from s_1) is assigned to a class between "Weakly bad" and "Weakly good" with respect to Literature, i.e. $s_2 \rightarrow_{Lit}$ [Weakly bad$_{Lit}$, Weakly good$_{Lit}$]. The compatibility condition relative to the assignment with respect to subcriterion $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, is as follows:

$$\forall a^*, b^* \in A^R, L_{\mathbf{r}}^{DM}(a^*) > R_{\mathbf{r}}^{DM}(b^*) \Rightarrow U_{\mathbf{r}}(a^*) > U_{\mathbf{r}}(b^*). \tag{4}$$

At the output, for each $a \in A$, besides the comprehensive possible assignments $C^P(a)$ and the necessary assignments $C^N(a)$, the method gives the possible assignment $C_{\mathbf{r}}^P(a)$ and the necessary assignment $C_{\mathbf{r}}^N(a)$ for each $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, as follows:

- $C_{\mathbf{r}}^P(a) = \left[L_{\mathbf{r},P}^{\mathcal{U}}(a), R_{\mathbf{r},P}^{\mathcal{U}}(a)\right] = \cup_{U \in \mathcal{U}} \left[L_{\mathbf{r}}^U(a), R_{\mathbf{r}}^U(a)\right]$,

- $C_{\mathbf{r}}^N(a) = \left[L^{\mathcal{U}}_{\mathbf{r},N}(a), R^{\mathcal{U}}_{\mathbf{r},N}(a)\right] = \cap_{U \in \mathcal{U}} \left[L_{\mathbf{r}}^U(a), R_{\mathbf{r}}^U(a)\right]$,

where

$$L_{\mathbf{r}}^U(a) = \max\left(\{1\} \cup \left\{L_{\mathbf{r}}^{DM}(a^*) : U_{\mathbf{r}}(a^*) \leq U_{\mathbf{r}}(a), a^* \in A^*\right\}\right),$$

$$R_{\mathbf{r}}^U(a) = \min\left(\{p\} \cup \left\{R_{\mathbf{r}}^{DM}(a^*) : U_{\mathbf{r}}(a^*) \geq U_{\mathbf{r}}(a), a^* \in A^*\right\}\right).$$

Group decision

In many decision making situations there is a plurality of DMs. For example, in case of decision related to land development, a group of stakeholders with different perceptions of predefined criteria has to be involved. ROR ([13, 9]) has been applied to group decision as follows. Considering a set $\mathcal{D}$ of DMs, and a set of pairwise comparisons provided by the DM belonging to $\mathcal{D}^{'} \subseteq \mathcal{D}$, for each DM $d_h \in \mathcal{D}^{'}$ we find the necessary and possible preference relations $\succsim_h^N$ and $\succsim_h^P$. Then, we can represent consensus between decision makers from $\mathcal{D}$, defining the following preference relations for all $\mathcal{D}' \subseteq \mathcal{D}$:

- the necessary-necessary preference relation ($\succsim_{\mathcal{D}'}^{N,N}$), for which a is necessarily preferred to b for all $d_h \in \mathcal{D}^{'}$,
- the necessary-possibly preference relation ($\succsim_{\mathcal{D}'}^{N,P}$), for which a is necessarily preferred to b for at least one $d_h \in \mathcal{D}^{'}$,
- the possibly-necessary preference relation ($\succsim_{\mathcal{D}'}^{P,N}$), for which a is possibly preferred to b for all $d_h \in \mathcal{D}^{'}$,
- the possibly-possibly preference relation ($\succsim_{\mathcal{D}'}^{P,P}$), for which a is possibly preferred to b for at least one $d_h \in \mathcal{D}^{'}$.

In case of the hierarchy of criteria we can define the above four relations for each subcriterion $G_{\mathbf{r}}$ from the hierarchy, excluding the elementary subcriteria, i.e. $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$.

Interacting criteria

UTA^{GMS}, UTADIS^{GMS} and GRIP take into account an additive value function. This model is among the most popular ones because it has the advantage of being easily manageable, and, moreover, it has a very sound axiomatic basis (see, e.g., [20, 27]). However, the additive value function is not able to represent *interactions* among criteria. For example, consider evaluation of cars using such criteria as maximum speed, acceleration and price. In this case, there may exist a negative interaction (*negative synergy*) between maximum speed and acceleration because a car with a high maximum speed also has a good acceleration, so, even if each of these two criteria is very important for a DM who likes sport cars, their joint impact on reinforcement of preference of a more speedy and better accelerating car over a less speedy and worse accelerating car will be smaller than a simple addition of the impacts of the two criteria considered separately in validation of this preference relation. In the same decision problem, there may exist a positive interaction (*positive synergy*) between maximum speed and price because a car with a high maximum speed is usually expensive, and thus a car with a high maximum speed and relatively low price is very much appreciated. Thus, the comprehensive impact of these two criteria on the strength of preference of a more speedy and cheaper car over a less speedy and more expensive car is greater than the impact of the two criteria considered separately in validation of this preference relation. To handle the interactions among criteria, one can consider *non-additive integrals*, such as Choquet integral [5] and Sugeno integral [26], or an additive value function augmented by additional components reinforcing the value when there is a positive interaction for some pairs of criteria, or penalizing the value when this interaction is negative, like in UTA^{GMS}-INT [15]. In case of the hierarchy of criteria we can consider interaction among criteria at each level of the hierarchy. For example, evaluating students we can have negative synergy (redundancy) for Mathematics and Physics (because, in general, good students in Mathematics are good also in Physics) and positive synergy for Algebra and Analysis at a lower level (because Algebra and Analysis require different aptitudes, and therefore a student good in Algebra is not always good in Analysis).

8 Conclusions

In this paper, in order to deal with one important issue of Multiple Criteria Decision Aiding (MCDA), that is the hierarchy of criteria, we proposed a new methodology called Multiple Criteria Hierarchy Process (MCHP). The basic idea of MCHP relies on consideration of preference relations regarding subcriteria at each level of the hierarchy of criteria, obtaining in this way a better insight into the problem at hand. MCHP can be applied to any MCDA method. In this paper, we considered the case where evaluations of alternatives are aggregated by a value function, and we applied MCHP to one particular MCDA methodology that is the Robust Ordinal Regression (ROR). In this case, the preference model is the entire set of general additive value functions compatible with preference information given by the Decision Maker (DM) in terms of pairwise comparisons of some alternatives, and in terms of intensity of preference with respect to some pairs of alternatives. The advantage is twofold:

- from the point of view of preference information, the hierarchy of criteria is enriching the possibility of the DM to express his/her preferences: in fact, the DM can give preference information at a comprehensive level, e.g., student s_1 is comprehensively preferred to student s_2, as well as at an intermediate level with respect to subcriteria, e.g., student s_1 is preferred to student s_2 on a subset of criteria related to Mathematics;

- with respect to decision support, taking into account the hierarchy of criteria permits to define possible and necessary preference relations not only at a comprehensive level but also at each intermediate level of the hierarchy: in fact, as a final result, we can have not only that student s_1 is comprehensively necessarily preferred to student s_2, and student s_3 is comprehensively possibly preferred to student s_4, but also that, e.g., student s_1 is necessarily preferred to student s_2 on a subset of criteria related to Mathematics, and s_3 is possibly preferred to student s_4 on criteria related to Organic Chemistry.

Adapting ROR to the hierarchy of criteria, i.e. putting together MCHP and ROR, gives a very powerful methodology of multiple criteria decision aiding: in fact, in this way we conjugate, on one hand, the robustness concerns by taking into account the set of all value functions compatible with preference information supplied by DM, and, on the other hand, the benefits of the hierarchical decomposition of a complex multiple criteria decision problem. We have shown, moreover, that all the methodological developments proposed within the ROR can be used in the case of the hierarchy of criteria: calculation of a representative value function, consideration of different credibilities of preference information, extreme ranking analysis, application to sorting problems, group decision, handling interaction among criteria. Let us observe that we can consider preference relations referring to a subset of criteria also if there is no explicit hierarchy in the set of criteria. In fact, for any subset of criteria $\mathcal{J}$, the DM can always express preferences of the type "a is preferred to b with respect to $\mathcal{J}$", as well as we can define necessary and possible preference relations with respect to $\mathcal{J}$.

We envisage three further methodological developments of the ROR adapted to the case of the hierarchy of criteria:

- consideration of imprecise evaluations on specific criteria;
- consideration of the outranking preference models;
- consideration of a structure of criteria more complex than the hierarchy defined in this paper: for example, while in this paper we assume that each subcriterion descends from only one criterion located at the upper level of the hierarchy tree, we can have a real situation where one subcriterion descends from more than one criterion of the upper level; for example, in case of evaluation of students at a scientific faculty, Analytic Mechanics can descend from both Mathematics and Physics; we also plan to deal with more complex criteria structures, like those considered in Analytical Network Process (ANP) [23].

Acknowledgment

The third author wishes to acknowledge financial support from the Polish National Science Centre, grant no. N N519 441939.

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Appendix

Proof of Proposition 4.2

1. For all $a, b \in A$

$$a \succsim_{\mathbf{r}}^{N} b \Leftrightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \Rightarrow \exists U \in \mathcal{U} : U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \Leftrightarrow a \succsim_{\mathbf{r}}^{P} b,$$

   thus we proved that $\succsim_{\mathbf{r}}^{N} \subseteq \succsim_{\mathbf{r}}^{P}$.

2. We have

$$\forall a \in A, \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(a) \Leftrightarrow \forall a \in A, \quad a \succsim_{\mathbf{r}}^{N} a,$$

   and therefore $\succsim_{\mathbf{r}}^{N}$ is reflexive.

   For all $a, b, c \in A$,

$$a \succsim_{\mathbf{r}}^{N} b, b \succsim_{\mathbf{r}}^{N} c \Leftrightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(c) \Rightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(c) \Rightarrow a \succsim_{\mathbf{r}}^{N} c$$

   and therefore $\succsim_{\mathbf{r}}^{N}$ is transitive. Being reflexive and transitive $\succsim_{\mathbf{r}}^{N}$ is a partial preorder.

3. For all $a, b \in A$,

$$a \not\succsim^P_{\mathbf{r}} b \Leftrightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) < U_{\mathbf{r}}(b) \Rightarrow \exists U \in \mathcal{U} : U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(a) \Leftrightarrow b \succsim^P_{\mathbf{r}} a$$

and therefore $\succsim^P_{\mathbf{r}}$ is strongly complete.

For all $a, b, c \in A$,

$$a \not\succsim^P_{\mathbf{r}} b \text{ and } b \not\succsim^P_{\mathbf{r}} c \Leftrightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) < U_{\mathbf{r}}(b) < U_{\mathbf{r}}(c) \Rightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) < U_{\mathbf{r}}(c) \Rightarrow a \not\succsim^P_{\mathbf{r}} c$$

and thus we proved that $\succsim^P_{\mathbf{r}}$ is negatively transitive.

4. For all $a, b \in A$,

$$a \not\succsim^N_{\mathbf{r}} b \Leftrightarrow \exists U \in \mathcal{U} : U_{\mathbf{r}}(a) < U_{\mathbf{r}}(b) \Rightarrow \exists U \in \mathcal{U} : U_{\mathbf{r}}(a) \leq U_{\mathbf{r}}(b) \Rightarrow b \succsim^P_{\mathbf{r}} a$$

and therefore we proved that $a \succsim^N_{\mathbf{r}} b$ or $b \succsim^P_{\mathbf{r}} a$.

5. For all $a, b, c \in A$, $a \succsim^N_{\mathbf{r}} b$ implies that $U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b)$ for all compatible value functions; $b \succsim^P_{\mathbf{r}} c$ implies that there exist at least one compatible value function $\overline{U}$ such that $\overline{U}_{\mathbf{r}}(b) \geq \overline{U}_{\mathbf{r}}(c)$; then for this compatible value function we have $\overline{U}_{\mathbf{r}}(a) \geq \overline{U}_{\mathbf{r}}(b) \geq \overline{U}_{\mathbf{r}}(c)$, and thus $a \succsim^P_{\mathbf{r}} c$.

6. $a \succsim^P_{\mathbf{r}} b$ implies that there exist at least one compatible value function $\overline{U}$ such that $\overline{U}_{\mathbf{r}}(a) \geq \overline{U}_{\mathbf{r}}(b)$; $b \succsim^N_{\mathbf{r}} c$ implies that $U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(c), \forall U \in \mathcal{U}$; in this way for the value function $\overline{U}$ we have $\overline{U}_{\mathbf{r}}(a) \geq \overline{U}_{\mathbf{r}}(b) \geq \overline{U}_{\mathbf{r}}(c)$, and thus $a \succsim^P_{\mathbf{r}} c$;

Proof of Proposition 4.3

1. Remembering that $U_{\mathbf{r}}(x) = U_{(\mathbf{r},1)}(x) + \ldots + U_{(\mathbf{r},n(\mathbf{r}))}(x)$, we have

$$a \succsim^N_{(\mathbf{r},j)} b \;\; \forall j = 1, \ldots, n(\mathbf{r}) \Leftrightarrow U_{(\mathbf{r},j)}(a) \geq U_{(\mathbf{r},j)}(b) \;\; \forall U \in \mathcal{U}, \forall j = 1, \ldots, n(\mathbf{r}) \Rightarrow$$
$$\Rightarrow \forall U \in \mathcal{U}, \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(b) \Leftrightarrow \forall U \in \mathcal{U}, \;\; U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \Leftrightarrow a \succsim^N_{\mathbf{r}} b.$$

2. $a \succsim^P_{(\mathbf{r},w)} b$ implies that there exists $\overline{U} \in \mathcal{U}$ such that $\overline{U}_{(\mathbf{r},w)}(a) \geq \overline{U}_{(\mathbf{r},w)}(b)$; considering that $a \succsim^N_{(\mathbf{r},j)} b$ for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{w\}$, we have $U_{(\mathbf{r},j)}(a) \geq U_{(\mathbf{r},j)}(b) \;\; \forall j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{w\}$, and therefore

also for $\overline{U} \in \mathcal{U}$ we have $\overline{U}_{(\mathbf{r},j)}(a) \geq \overline{U}_{(\mathbf{r},j)}(b) \ \ \forall j \in \{1,\ldots,n(\mathbf{r})\} \setminus \{w\}$ and thus

$$\overline{U}_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(b) = \overline{U}_{\mathbf{r}}(b),$$

from which $a \succsim_{\mathbf{r}}^{P} b$.

3. Let us suppose, for contradiction, that $a \succsim_{\mathbf{r}}^{P} b$; this means that there exists a value function $\overline{U} \in \mathcal{U}$ such that $\overline{U}_{\mathbf{r}}(a) \geq \overline{U}_{\mathbf{r}}(b)$; from this we obtain that

$$\overline{U}_{\mathbf{r}}(a) \geq \overline{U}_{\mathbf{r}}(b) \Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(b) \Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \left[\overline{U}_{(\mathbf{r},j)}(a) - \overline{U}_{(\mathbf{r},j)}(b)\right] \geq 0$$

and from this, for at least one $j \in \{1,\ldots,n(\mathbf{r})\}$ we have $\overline{U}_{(\mathbf{r},j)}(a) - \overline{U}_{(\mathbf{r},j)(b)} \geq 0 \Rightarrow \overline{U}_{(\mathbf{r},j)}(a) \geq \overline{U}_{(\mathbf{r},j)}(b)$ and thus $a \succsim_{(\mathbf{r},j)}^{P} b$ which contradicts the hypothesis.

Proof of Proposition 5.1

Let us remember that $\forall a \in A$ we have $U_{\mathbf{r}}(a) = U_{(\mathbf{r},1)}(a) + \ldots + U_{(\mathbf{r},n(\mathbf{r}))}(a)$.

1. For any $a,b,c,d \in A$

$$(a,b) \succsim_{(\mathbf{r},j)}^{*N} (c,d) \ \ \forall j = 1,\ldots,n(\mathbf{r}) \Leftrightarrow$$

$$\Leftrightarrow U_{(\mathbf{r},j)}(a) - U_{(\mathbf{r},j)}(b) \geq U_{(\mathbf{r},j)}(c) - U_{(\mathbf{r},j)}(d), \forall U \in \mathcal{U}, \forall j = 1,\ldots,n(\mathbf{r}) \Rightarrow$$

$$\Rightarrow \forall U \in \mathcal{U}, \sum_{j=1}^{n(\mathbf{r})} \left[U_{(\mathbf{r},j)}(a) - U_{(\mathbf{r},j)}(b)\right] \geq \sum_{j=1}^{n(\mathbf{r})} \left[U_{(\mathbf{r},j)}(c) - U_{(\mathbf{r},j)}(d)\right] \Leftrightarrow$$

$$\Leftrightarrow \forall U \in \mathcal{U}, \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) - \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(b) \geq \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(c) - \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(d) \Leftrightarrow$$

$$\Leftrightarrow \forall U \in \mathcal{U}, U_{\mathbf{r}}(a) - U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(c) - U_{\mathbf{r}}(d) \Leftrightarrow (a,b) \succsim_{\mathbf{r}}^{*N} (c,d).$$

2. For any $a,b,c,d \in A$, $(a,b) \succsim_{(\mathbf{r},w)}^{*P} (c,d)$ implies that there exists $\overline{U} \in \mathcal{U}$ such that $\overline{U}_{(\mathbf{r},w)}(a) - \overline{U}_{(\mathbf{r},w)}(b) \geq \overline{U}_{(\mathbf{r},w)}(c) - \overline{U}_{(\mathbf{r},w)}(d)$; considering that $(a,b) \succsim_{(\mathbf{r},j)}^{*N} (c,d)$ for all $j \in \{1,\ldots,n(\mathbf{r})\} \setminus \{w\}$ and for all compatible value functions, we have $\overline{U}_{(\mathbf{r},j)}(a) - \overline{U}_{(\mathbf{r},j)}(b) \geq \overline{U}_{(\mathbf{r},j)}(c) - \overline{U}_{(\mathbf{r},j)}(d) \ \ \forall j \in \{1,\ldots,n(\mathbf{r})\} \setminus \{w\}$, and thus

$$\sum_{j=1}^{n(\mathbf{r})} \left[\overline{U}_{(\mathbf{r},j)}(a) - \overline{U}_{(\mathbf{r},j)}(b)\right] \geq \sum_{j=1}^{n(\mathbf{r})} \left[\overline{U}_{(\mathbf{r},j)}(c) - \overline{U}_{(\mathbf{r},j)}(d)\right] \Leftrightarrow$$

$$\Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) - \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(b) \geq \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(c) - \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(d) \Leftrightarrow$$

$$\Leftrightarrow \overline{U}_{\mathbf{r}}(a) - \overline{U}_{\mathbf{r}}(b) \geq \overline{U}_{\mathbf{r}}(c) - \overline{U}_{\mathbf{r}}(d) \Leftrightarrow (a,b) \succsim_{\mathbf{r}}^{*^P} (c,d).$$

from which $(a,b) \succsim_{\mathbf{r}}^{*^P} (c,d)$.

3. Let us suppose, for contradiction, that for $a,b,c,d \in A$ $(a,b) \succsim_{\mathbf{r}}^{*^P} (c,d)$; this means that there exists a value function $\overline{U} \in \mathcal{U}$ such that $\overline{U}_{\mathbf{r}}(a) - \overline{U}_{\mathbf{r}}(b) \geq \overline{U}_{\mathbf{r}}(c) - \overline{U}_{\mathbf{r}}(d)$; from this we obtain that

$$\overline{U}_{\mathbf{r}}(a) - \overline{U}_{\mathbf{r}}(b) \geq \overline{U}_{\mathbf{r}}(c) - \overline{U}_{\mathbf{r}}(d) \Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) - \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(b) \geq \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(c) - \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(d) \Leftrightarrow$$

$$\Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) - \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(b) - \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(c) + \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(d) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \left[\overline{U}_{(\mathbf{r},j)}(a) - \overline{U}_{(\mathbf{r},j)}(b) + \overline{U}_{(\mathbf{r},j)}(c) - \overline{U}_{(\mathbf{r},j)}(d)\right] \geq 0,$$

and from this follows that, for at least one $j \in \{1, \ldots, n(\mathbf{r})\}$ we have $\overline{U}_{(\mathbf{r},j)}(a) - \overline{U}_{(\mathbf{r},j)}(b) \geq \overline{U}_{(\mathbf{r},j)}(c) - \overline{U}_{(\mathbf{r},j)}(d)$, and thus $(a,b) \succsim_{(\mathbf{r},j)}^{*^P} (c,d)$ for at least one j, which contradicts the hypothesis.