Multiple Criteria Hierarchy Process with ELECTRE and PROMETHEE

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Abstract: Robust Ordinal Regression (ROR) supports Multiple Criteria Decision Process by considering all sets of parameters of an assumed preference model, that are compatible with preference information elicited by a Decision Maker (DM). As a result of ROR, one gets necessary and possible preference relations in the set of alternatives, which hold for all compatible sets of parameters, or for at least one compatible set of parameters, respectively. In this paper, we propose an extension of ELECTRE and PROMETHEE methods to the case of the hierarchy of criteria, which was never considered before. Then, we adapt ROR to the hierarchical versions of ELECTRE and PROMETHEE methods.

Keywords: Hierarchy of criteria, ELECTRE methods, PROMETHEE methods, Robust Ordinal Regression

1 Introduction

Multiple Criteria Decision Aiding (MCDA) copes with three main types of decision problems: ranking, sorting and choice. Ranking problems consist of rank ordering of all alternatives from the worst to the best, looking at their evaluations on the considered criteria; sorting problems consist in assigning each alternative to a predefined and preference ordered class; choice problems consist in selecting a subset of alternatives considered as the best (for a more detailed survey, see [6, 4]). In order to handle these problems, one can use either of the two different methodologies:

- assign to each alternative a utility value, i.e. a real number reflecting the degree of desirability of a considered alternative, independently from the evaluations of other alternatives,
- compare alternatives pairwise, in order to discover if one is preferred to the other, or if they are indifferent or incomparable.

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In the first case, to associate a utility value to an alternative, taking into account its evaluations on the considered criteria, multi attribute utility theory (MAUT) [14] frequently uses an additive value function defined as a sum of as many marginal value functions as there are criteria. In the second case, outranking methods [2, 16] construct a binary relation which reads: "alternative a is at least as good as alternative b", which means "a outranks b". This construction takes into account evaluation of both compared alternatives on the considered criteria, as well as some comparison thresholds and weights expressing the relative importance of the criteria.

Generally, the information provided by the dominance relation on the set of alternatives is poor, and makes many alternatives incomparable. To enrich this relation, the Decision Maker (DM) is asked to provide some preference information, so that the outranking relation giving account of it makes alternatives more comparable. As this comparability is consistent with the value system of the DM, the outranking relation can be considered as the DM's preference model.

Preference information can be direct or indirect; direct means that the DM can give information regarding values of parameters of the considered preference model, while indirect means that the DM gives information regarding some alternatives (s)he knows well, and from this information there are inferred values of parameters of the considered preference model. Generally, the indirect methodology is more realistic (see, e.g.,[12],[15],[20]), because the DM does not always understand well enough the meaning of all these parameters. Using the indirect methodology, there usually exist more than one set of parameters compatible with the preference information provided by the DM, and each of these sets of parameters could give different results to the decision problem at hand. For this reason, any choice of one specific set of parameters compatible with preference information provided by the DM could be considered as arbitrary and meaningless. In order to deal with this inconvenience, Robust Ordinal Regression (ROR) takes into account not only one set of parameters compatible with the preference information provided by the DM, but considers all these sets simultaneously defining two preference relations:

- the necessary preference relation, for which "alternative a is necessarily preferred to alternative b" if a is at least as good as b for all compatible sets of parameters,
- the possible preference relation, for which "alternative a is possibly preferred to alternative b" if a is at least as good as b for at least one compatible set of parameters.

ROR methods have been proposed for ranking and choice problems [8, 10], sorting problems [11], outranking models [9] and non additive models [1].

Remark that not all multiple criteria decision problems present evaluation criteria at the same level, but there can exist a hierarchical structure of criteria. This is the case, for example, of environmental planning in which it is possible to take into account economic, social and environmental criteria, and each of these criteria can be composed of subcriteria on which the alternatives are evaluated. In [3], we have considered the hierarchy of criteria in the context of ROR, showing the following advantages of using this procedure:

- the DM can express preference information not only in a comprehensive way but also in a partial way, that is considering preference information with respect to a subcriterion at an intermediate level of the hierarchy,
- the DM can obtain results not only with respect to the comprehensive view, but also results at intermediate levels of the hierarchy; for example, the DM can learn if a is necessarily or possibly preferred to b with respect to a subcriterion G at an intermediate level of the hierarchy.

Let us remark that the use of the hierarchy of criteria proposed by our approach is rather different from other MCDA methodologies assuming a hierarchical structure of the family of criteria. In fact, while in general the hierarchy of criteria is used to decompose and make easier the preference elicitation concerning pairwise comparisons of criteria with respect to relative importance, in our approach, a preference relation in each node of the hierarchy constitutes a base for the discussion with the DM.

Indeed, the preference relations in particular nodes of the hierarchy are presented to the DM as consequences (output) of her/his preference information provided at the input. In course of an interactive process, the DM can add, modify or remove some items of the preference information if (s)he feels that the preference relations do not reflect correctly her/his value system. This interactive process ends when the DM gets convinced by the preference relations obtained in consequence of her/his preference information, and thus accepts the recommendations provided by the MCDA methodology.

Observe that consideration of preference relations at each level of the hierarchy constitutes a specific feature of our methodology, which we consider very useful in any decision process in which a hierarchy of criteria is considered. Considering preference relations in particular nodes of the hierarchy permits decomposition of arguments explaining the overall preferences. For example, in case of evaluations of students, one could say that student a is comprehensively preferred to student b, because even if a is slightly worse than b with respect to subjects related to Literature, he is much better with respect to subjects related to Mathematics and Physics. Moreover, going in depth of the hierarchy, one could add that the preference with respect to subjects related to Mathematics is based on better evaluations of student a on subjects related to Analysis rather than on subjects related to Algebra.

It is worth noting that this specific use of the hierarchy of criteria can be applied to any MCDA methodology. In this paper, we are applying it to Robust Ordinal Regression approach, but it can be applied to any other MCDA methodology, even those which use the hierarchy to ask the DM for pairwise comparisons of subcriteria with respect to their importance.

In this paper, we propose a generalization of outranking methods, more specifically ELECTRE and PROMETHEE methods, to the case of the hierarchy of criteria. No similar attempt is known in the literature. We extend the methodologies of ELECTRE and PROMETHEE to the case where the considered criteria are not at the same level, but they are structured into several levels. In this way, the DM can obtain information not only regarding the comprehensive outranking of an alternative a over an alternative b, but also partially, that is, considering a particular criterion/subcriterion of the hierarchy. For example, in the environmental planning problem, it will be possible to investigate if a certain location p_1 outranks another location p_2 with respect to economic criteria, or environmental criteria, or social aspects, or with respect to all criteria simultaneously. In this particular context, it is worth stressing that ELECTRE and PROMETHEE methods can be considered as particular cases of our methodology, and for this reason, it can be considered as a real generalization of these methods.

In the perspective of considering a constructive interaction between the DM and the analyst, we intend to use the ROR methodology to deal with outranking methods in case of the hierarchy of criteria. The application of ROR to ELECTRE and PROMETHEE methods has already been done in [9] and [13], respectively, but also in this case, our methodology can be considered as their generalization because, of course, the absence of hierarchy corresponds to the case of a hierarchy with only one level containing all the criteria.

The paper is organized in the following way: in section 2, we recall the principal concept of the hierarchy of criteria and describe the ELECTRE method generalized to this case; in section 3, we extend the concept of ROR applied to ELECTRE (which constitutes ELECTRE^{GKMS} method) in case of the hierarchy of criteria, and we propose a didactic example illustrating the use of ELECTRE and ELECTRE^{GKMS} methods applied to a hierarchical structure of criteria; in section 4, we extend the PROMETHEE method to the case of the hierarchy of criteria; in section 5, we describe the application of ROR to the PROMETHEE method in case of the hierarchy of criteria, and we provide an example illustrating the PROMETHEE and PROMETHEE^{GKS} methods applied to a hierarchy of criteria; section 6 collects conclusions.

2 Hierarchical ELECTRE method

In this section, we recall the basic concepts of the hierarchy of criteria introduced in [3], and we introduce the Hierarchical ELECTRE method.

2.1 Hierarchical structure of the set of criteria

We suppose that evaluation criteria are not at the same level but they are structured into several levels (see Figure 1);

- $A = \{a, b, c, \ldots\}$ is the finite set of alternatives,
- l is the number of levels in the hierarchy of criteria,
- $\mathcal{G}$ is the set of all criteria at all considered levels,
- $\mathcal{I}_{\mathcal{G}}$ is the set of indices of particular criteria representing position of the criteria in the hierarchy,
- m is the number of the first level (root) criteria, $G_{\mathbf{1}}, \ldots, G_{\mathbf{m}}$,
- $G_{\mathbf{r}} \in \mathcal{G}$, with $\mathbf{r} = (i_1, \ldots, i_h) \in \mathcal{I}_{\mathcal{G}}$, denotes a subcriterion of the first level criterion G_{i_1} at level h,
- $n(\mathbf{r})$ is the number of subcriteria of $G_{\mathbf{r}}$ in the subsequent level, i.e. the direct subcriteria of $G_{\mathbf{r}}$ are $G_{(\mathbf{r},1)}, \ldots, G_{(\mathbf{r},n(\mathbf{r}))}$,
- $g_{\mathbf{t}} : A \rightarrow \mathbb{R}$, with $\mathbf{t} = (i_1, \ldots, i_l) \in \mathcal{I}_{\mathcal{G}}$, denotes an elementary subcriterion of the first level criterion G_{i_1}, i.e a subcriterion at level l,
- EL is the set of indices of all elementary subcriteria:

$$EL = \{\mathbf{t} = (i_1, \ldots, i_l) \in \mathcal{I}_{\mathcal{G}}\}, \quad \text{where} \begin{cases} i_1 = 1, \ldots, m \\ i_2 = 1, \ldots, n(i_1) \\ \ldots\ldots \\ i_l = 1, \ldots, n(i_1, \ldots, i_{l-1}) \end{cases}$$

- $E(G_{\mathbf{r}})$ is the set of indices of elementary subcriteria descending from $G_{\mathbf{r}}$, i.e.

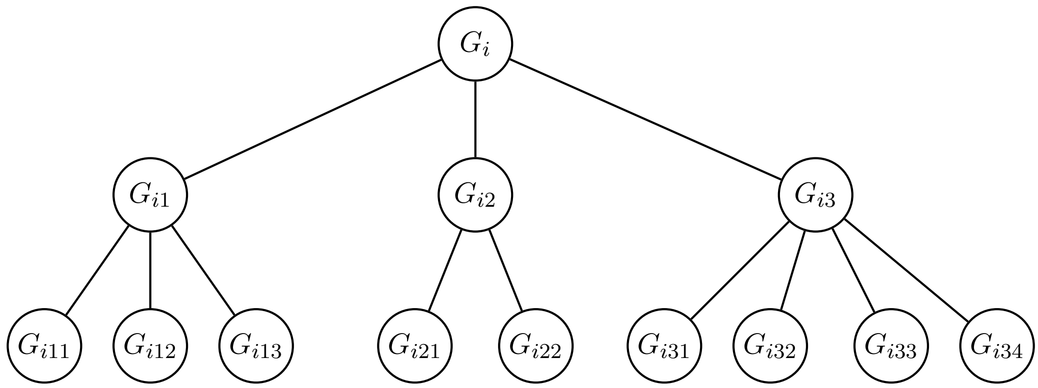
$$E(G_{\mathbf{r}}) = \{(\mathbf{r}, i_{h+1}, \ldots, i_l) \in \mathcal{I}_{\mathcal{G}}\}, \quad \text{where} \begin{cases} i_{h+1} = 1, \ldots, n(\mathbf{r}) \\ \ldots\ldots \\ i_l = 1, \ldots, n(\mathbf{r}, i_{h+1}, \ldots, i_{l-1}) \end{cases}$$

  thus $E(G_{\mathbf{r}}) \subseteq EL$ and, more precisely, $E(G_r) = EL$ if all elementary subcriteria descend from criterion $G_{\mathbf{r}}$,

- LBO is the set of indices of all subcriteria of the last but one level,

- $LB(G_{\mathbf{r}})$ is the set of indices of subcriteria of the last but one level descending from criterion/subcriterion $G_{\mathbf{r}}$,
- when $\mathbf{r} = 0$, then by $G_{\mathbf{r}} = G_{\mathbf{0}}$, we mean the entire set of criteria and not a particular criterion or subcriterion; in this particular case, we have $E(G_{\mathbf{0}}) = EL$ and $LB(G_{\mathbf{0}}) = LBO$.

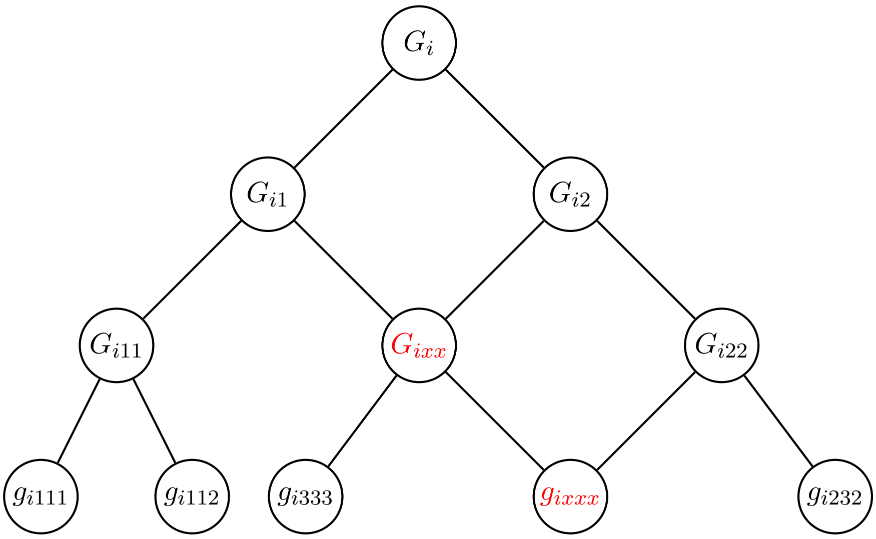
Figure 1: Example of the hierarchy of criteria starting from the first level (root) criterion G_i



Remark that, without loss of generality, we consider a hierarchical structure where each criterion belongs to only one criterion of the level immediately above, that is a criterion $G_{\mathbf{r}}$ from the i-th level of the hierarchy is a subcriterion of only one of the criteria of the $(i-1)$-th level (we call a structure of this type a partitioned structure). Example of the hierarchy of criteria with a partitioned structure is presented in Figure 1. In order to understand the reason of this restriction, let us examine an example of the hierarchy of criteria with a non-partitioned structure shown in Figure 2. In this particular structure, criterion G_{ixx} and elementary subcriterion g_{ixxx} are subcriteria of more than one criterion of the level immediately above. In particular, criterion G_{ixx} is a subcriterion of criteria G_{i1} and G_{i2} while elementary subcriterion g_{ixxx} is a subcriterion of criteria G_{ixx} and G_{i22}. This means that both criteria influence the criteria they descend from, in a different way. That is, the evaluations of alternatives with respect to elementary subcriterion g_{ixxx} will be weighted in one way if g_{ixxx} is considered to be a subcriterion of criterion G_{ixx}, and they could be weighted in another way if g_{ixxx} is considered to be a subcriterion of criterion G_{i22}. In this way, we can distinguish the contribution of g_{ixxx} to G_{ixx}, from the contribution of g_{ixxx} to G_{i22}. In order to take into account these different types of contribution, we propose to split g_{ixxx} in two "indicators": g_{ixx2}, representing the contribution of elementary subcriterion g_{ixxx} to G_{ixx}, and g_{i221}, representing the contribution of elementary subcriterion g_{ixxx} to criterion G_{i22}. All alternatives will keep the same evaluations with respect to indicators g_{ixx2} and g_{i221}, as they had with respect to criterion g_{ixxx}, but their weights k_{ixx2} and k_{i221} could be different and, moreover, they have to satisfy the relation: $k_{ixx2} + k_{i221} = k_{ixxx}$; this means that the sum of weights of new indicators g_{i221} and g_{ixxx} has to be equal to the weight of criterion g_{ixxx}. Doing in this way, we obtain the hierarchical structure shown in Figure 3. At this point, we observe that G_{ixx} is a subcriterion in common of G_{i1} and G_{i2}, thus, we may proceed analogically to g_{ixxx}. So, we have to distinguish between the contribution of G_{ixx} to G_{i1} and the contribution of G_{ixx} to G_{i2}. But, in this case, subcriterion G_{ixx} influences the two above criteria via all its subcriteria (if any) and elementary subcriteria; for this reason we have to split all subcriteria and elementary subcriteria descending from it (here: g_{ixx1} and g_{ixx2}) in order to take into account the different contribution they give to the above criteria. In this way we obtain the partitioned hierarchical structure shown in Figure 4, where:

- subcriterion G_{ixx} is split into indicators G_{i12} and G_{i21},
- indicator g_{ixx1} is split into indicators g_{i121} and g_{i211}, and thus, $k_{ixx1} = k_{i121} + k_{i211}$,
- indicator g_{ixx2} is split into indicators g_{i122} and g_{i212}, and thus, $k_{ixx2} = k_{i122} + k_{i212}$.

Figure 2: Example of the hierarchy of criteria with a non-partitioned structure



Let us remark that the above splitting of criteria, which aims to take into account their contribution to different criteria at an upper level, can be applied independently of the type of the preference model used in the hierarchical MCDA method. Thus, it could also be used in the hierarchical method involving multiattribute utility functions presented in [3].

2.2 Handling the hierarchy of criteria in ELECTRE methods

In this sub-section, we introduce a generalization of ELECTRE methods to the case of the hierarchy of criteria. We start with the hierarchical generalization of the ELECTRE IS method, and then we extend this

Figure 3: Transformation of non-partitioned structure (step 1)

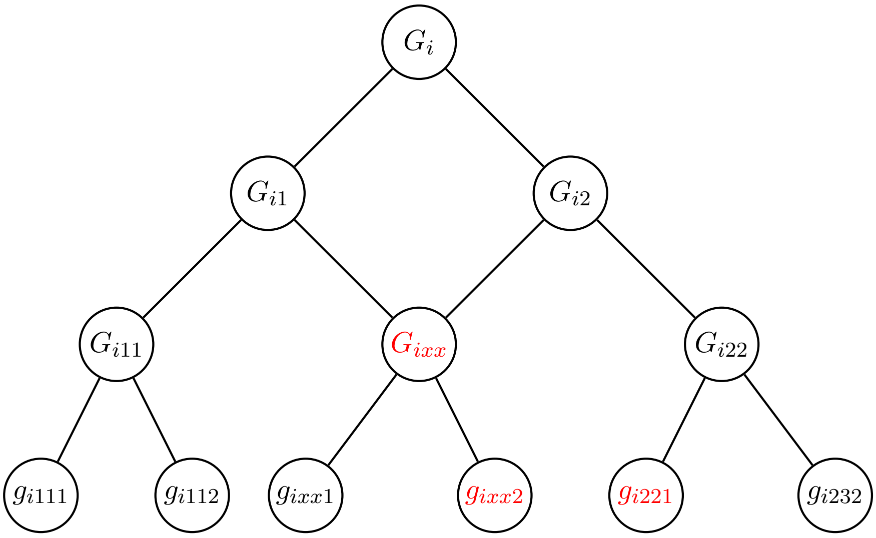
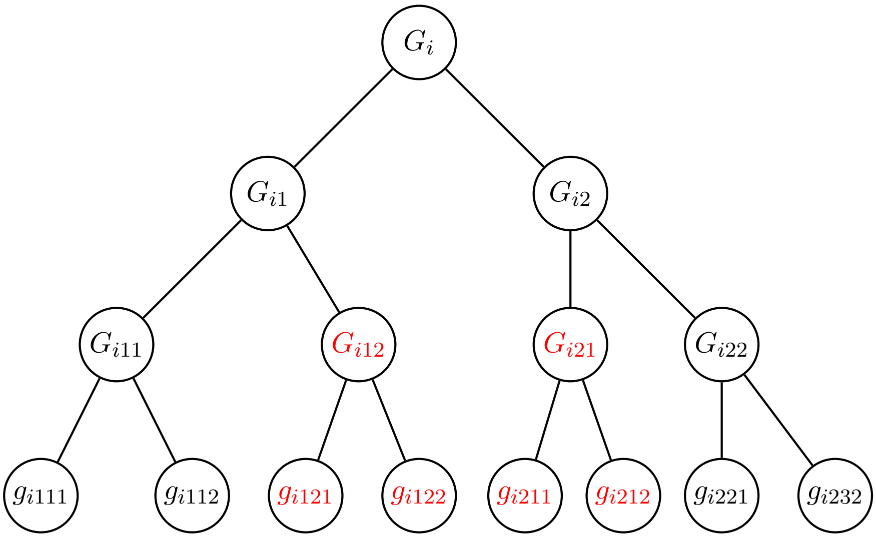


Figure 4: Partitioned structure resulting from the transformation of the non-partitioned one (step 2)



generalization on the ELECTRE III method (see [17, 18] for description of different ELECTRE methods). Given a criterion/subcriterion $G_{\mathbf{r}}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, an outranking relation is a binary relation $S_{\mathbf{r}} \subseteq A \times A$ (in the following $A \times A = B$), such that $aS_{\mathbf{r}}b$ means "a is at least as good as b with respect to criterion $G_{\mathbf{r}}$". Knowing if $S_{\mathbf{r}}$ is true or not for an ordered pair of alternatives $(a,b) \in B$, one is able to represent situations of *weak* ($Q_{\mathbf{r}}$) or *strict* ($P_{\mathbf{r}}$) *preference* (the two relations together called *large preference*), *indifference* ($\sim_{\mathbf{r}}$), and *incomparability* ($R_{\mathbf{r}}$) among a and b:

$$\left.\begin{array}{l} aS_{\mathbf{r}}b \text{ and } not(bS_{\mathbf{r}}a) \Leftrightarrow aQ_{\mathbf{r}}b \text{ or } aP_{\mathbf{r}}b, \\[2ex] aS_{\mathbf{r}}b \text{ and } bS_{\mathbf{r}}a \Leftrightarrow a \sim_{\mathbf{r}} b, \\[2ex] not(aS_{\mathbf{r}}b) \text{ and } not(bS_{\mathbf{r}}a) \Leftrightarrow aR_{\mathbf{r}}b. \end{array}\right\}$$

Let us denote by $k_{\mathbf{t}}$ the *weight* assigned to elementary subcriterion $g_{\mathbf{t}}, \mathbf{t} \in EL$. It is a non-negative real number representing the relative importance (strength) of elementary subcriterion $g_{\mathbf{t}}$ within the family of elementary subcriteria. The *indifference, preference*, and *veto thresholds* on elementary subcriterion $g_{\mathbf{t}}$ are denoted by $q_{\mathbf{t}}$, $p_{\mathbf{t}}$ and $v_{\mathbf{t}}$, respectively. $q_{\mathbf{t}}$ is the greatest difference between the evaluations of two alternatives, compatible with the indifference among them with respect to elementary subcriterion $g_{\mathbf{t}}$; $p_{\mathbf{t}}$ is the smallest difference between the evaluations of two alternatives, compatible with the preference of an alternative over another with respect to elementary subcriterion $g_{\mathbf{t}}$; $v_{\mathbf{t}}$ is an upper bound beyond which the discordance about the assertion "a is at least as good as b" cannot surpass. For consistency, $v_{\mathbf{t}} > p_{\mathbf{t}} \geq q_{\mathbf{t}} \geq 0$, for all $\mathbf{t} \in EL$. The thresholds on particular elementary subcriterion can be either constant for all alternatives, or dependent on evaluation of a, $g_{\mathbf{t}}(a)$. In the sequel, we assume, for simplicity, constant thresholds, although this is not a necessary assumption for our methodology. Moreover, we assume without loss of generality that all criteria are increasing monotone with respect to the preference, i.e. the greater the evaluation, the better it is.

Construction of an outranking relation involves two concepts known as *concordance* and *non-discordance* tests. The concordance test involves calculation of concordance index $C_{\mathbf{r}}(a, b)$. It represents the strength of the coalition of elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in E(G_{\mathbf{r}})$, being in favor of $aS_{\mathbf{r}}b$. This coalition is composed of two subsets of elementary subcriteria:

- subset of elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in E(G_{\mathbf{r}})$, being clearly in favor of $aS_{\mathbf{r}}b$, i.e. such that, $g_{\mathbf{t}}(a) \geq g_{\mathbf{t}}(b) - q_{\mathbf{t}}$,
- subset of elementary subcriteria $g_{\mathbf{t}}, \mathbf{t} \in E(G_{\mathbf{r}})$, that do not oppose to $aS_{\mathbf{r}}b$, while being in an ambiguous position with respect to this assertion, i.e. those with $bQ_{\mathbf{r}}a$, which is equivalent to $g_{\mathbf{t}}(b) - p_{\mathbf{t}} < g_{\mathbf{t}}(a) < g_{\mathbf{t}}(b) - q_{\mathbf{t}}$.

Note that $aS_{\mathbf{r}}b$ is true not only when alternative a is preferred to alternative b on criterion/subcriterion $G_{\mathbf{r}}$ but also when a is indifferent to b on $G_{\mathbf{r}}$, and even when b dominates a on $G_{\mathbf{r}}$ by a sufficiently small amount in each elementary subcriterion descending from $G_{\mathbf{r}}$.
Consequently, the partial concordance index is defined as:

$$C_{\mathbf{r}}(a,b) = \sum_{\mathbf{t}\in E(G_{\mathbf{r}})} \phi_{\mathbf{t}}(a,b) \times k_{\mathbf{t}} = \sum_{\mathbf{t}\in E(G_{\mathbf{r}})} \psi_{\mathbf{t}}(a,b) \tag{1}$$

where, traditionally, for each $\mathbf{t} \in E(G_{\mathbf{r}})$,

$$\phi_{\mathbf{t}}(a,b) = \begin{cases} 1, & \text{if } g_{\mathbf{t}}(a) \geq g_{\mathbf{t}}(b) - q_{\mathbf{t}}, \\ \frac{g_{\mathbf{t}}(a)-[g_{\mathbf{t}}(b)-p_{\mathbf{t}}]}{p_{\mathbf{t}}-q_{\mathbf{t}}}, & \text{if } g_{\mathbf{t}}(b) - p_{\mathbf{t}} \leq g_{\mathbf{t}}(a) < g_{\mathbf{t}}(b) - q_{\mathbf{t}}, \\ 0, & \text{if } g_{\mathbf{t}}(a) < g_{\mathbf{t}}(b) - p_{\mathbf{t}}. \end{cases} \tag{2}$$

$\phi_{\mathbf{t}}(a,b)$ is a marginal concordance index, indicating to what extent elementary subcriterion $g_{\mathbf{t}}, \mathbf{t} \in E(G_{\mathbf{r}})$, contributes to the concordance index $C_{\mathbf{r}}(a,b)$. In order to simplify calculations, and without loss of generality, we assume that the weights of elementary subcriteria sum up to one, i.e. $\sum_{\mathbf{t}\in EL} k_{\mathbf{t}} = 1$.

Note 2.1. *When comparing two alternatives a, b on a given elementary subcriterion, the zone between $-p_{\mathbf{t}}$ and $-q_{\mathbf{t}}$ corresponds to hesitation between opting for indifference and preference. In order to take into account this ambiguity, ELECTRE methods consider $\phi_{\mathbf{t}}(a,b)$ being linear and non-decreasing functions with respect to the difference $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b)$. The assumption of linearity of functions $\phi_{\mathbf{t}}(a,b)$ is only conventional and it is not related in any case to the concept of intensity of preference. Moreover, according to [7], slight changes of the form of $\phi_{\mathbf{t}}(a,b)$ have no impact (apart from very particular cases) on the results.*

Remark that $C_{\mathbf{r}}(a,b) \in [0, K_{\mathbf{r}}]$, where $K_{\mathbf{r}} = \sum_{\mathbf{t}\in E(G_{\mathbf{r}})} k_{\mathbf{t}}$, and $C_{\mathbf{r}}(a,b) = 0$ if $g_{\mathbf{t}}(a) \leq g_{\mathbf{t}}(b) - p_{\mathbf{t}}$, for all $\mathbf{t} \in E(G_{\mathbf{r}})$ (b is strictly preferred to a on all elementary subcriteria descending from $G_{\mathbf{r}}$), and $C_{\mathbf{r}}(a,b) = K_{\mathbf{r}}$ if $g_{\mathbf{t}}(a) \geq g_{\mathbf{t}}(b) - q_{\mathbf{t}}$, for all $\mathbf{t} \in E(G_{\mathbf{r}})$ (a outranks b on all elementary subcriteria descending from $G_{\mathbf{r}}$). When $\mathbf{r} = 0$, $C_{\mathbf{0}}(a,b) \in [0,1]$ because $E(G_{\mathbf{0}}) = EL$ and thus $K_{\mathbf{0}} = 1$.

In ELECTRE, the result of the concordance test concerning a pair of alternatives is positive when the value of the concordance index is not smaller than a fixed concordance cutting level. In the hierarchical extension of ELECTRE, we admit one concordance cutting level $\lambda_{\mathbf{r}}$ for each criterion/subcriterion $G_{\mathbf{r}}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, that is, we consider one concordance cutting level for each criterion/subcriterion except for elementary subcriteria, such that:

- $\lambda_{\mathbf{s}} \in [K_{\mathbf{s}}/2, K_{\mathbf{s}}]$, for all $\mathbf{s} \in LBO$,
- $\lambda_{\mathbf{r}} = \sum_{j=1}^{n(\mathbf{r})} \lambda_{(\mathbf{r},j)}$, for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$.

In particular, the first condition means that each concordance cutting level $\lambda_{\mathbf{s}}, \mathbf{s} \in LBO$, is bounded between the half-sum and the sum of the weights of elementary subcriteria descending from $G_{\mathbf{s}}$; the second condition means that the concordance cutting level of a criterion/subcriterion $G_{\mathbf{r}}$, is equal to the sum of the concordance cutting levels of subcriteria $G_{(\mathbf{r},\mathbf{j})}, j = 1, \ldots, n(\mathbf{r})$, at the level immediately below; we add this condition in order to avoid the case where an alternative a outranks an alternative b with respect to all subcriteria $G_{(\mathbf{r},1)}, \ldots, G_{(\mathbf{r},n(\mathbf{r}))}$ from the level immediately below $G_{\mathbf{r}}$, but a does not outrank b with respect to criterion/subcriterion $G_{\mathbf{r}}$. For example, it is obvious that if student s_1 outranks student s_2 with respect to Algebra and Analysis, being immediate subcriteria of Mathematics, then s_1 outranks s_2 also with respect to Mathematics.

Note 2.2. *Remark that the two above conditions ensure that:*

$$\lambda_{\mathbf{r}} \in \left[\frac{K_{\mathbf{r}}}{2}, K_{\mathbf{r}}\right], \quad \textit{for all } \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL.$$

This implies that not only the concordance cutting level of criteria from the last but one level of the hierarchy, but all concordance cutting levels $\lambda_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, are constrained between the half-sum and the sum of weights of elementary subcriteria descending from $G_{\mathbf{r}}$.

Note 2.3. *In case the DM is not confident in providing a concordance cutting level for each criterion belonging to the last but one level of the hierarchy, (s)he could give another information regarding them. In fact, (s)he could state that each concordance cutting level $\lambda_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, should be equal to a certain percentage of the sum of weights of the elementary subcriteria descending from criterion $G_{\mathbf{r}}$. For example, if the DM declared that the concordance cutting levels should be equal to 70% of the relative weights of elementary subcriteria descending from the corresponding criteron, it would give $\lambda_{\mathbf{r}} = 0.7 \times \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} k_{\mathbf{t}}$. It is easy to observe that also in this case $\lambda_{\mathbf{r}} = \sum_{j=1}^{n(\mathbf{r})} \lambda_{(\mathbf{r},j)}$ for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$.*

The result of the concordance test for a pair $(a, b) \in B$ is positive if $C_{\mathbf{r}}(a, b) \geq \lambda_{\mathbf{r}}$. Once the result of the concordance test has been positive, one can pass to the non-discordance test. Its result is positive for the pair $(a, b) \in B$, unless "a is significantly worse than b" on at least one elementary subcriterion descending from $G_{\mathbf{r}}$, i.e. if $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq v_{\mathbf{t}}$ for some $\mathbf{t} \in E(G_{\mathbf{r}})$.

Remark that, if we consider $\mathbf{r} = 0$ then this procedure boils down to the classical ELECTRE method in which all evaluation criteria are considered at the same level.

Summing up, for each criterion/subcriterion $G_{\mathbf{r}}$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and for each $a, b \in A$, we have:

$$aS_{\mathbf{r}}b \Leftrightarrow C_{\mathbf{r}}(a, b) \geq \lambda_{\mathbf{r}}, \text{ and } g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) < v_{\mathbf{t}}, \text{ for all } \mathbf{t} \in E(G_{\mathbf{r}}).$$

In the following Proposition we show two fundamental properties of hierarchical outranking:

Proposition 2.1.

1. *Given two alternatives $a, b \in A$, and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (LBO \cup EL)$, such that*

$$aS_{(\mathbf{r},j)}b, \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aS_{\mathbf{r}}b$,

2. *Given two alternatives $a, b \in A$, and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (LBO \cup EL)$, such that*

$$not(aS_{(\mathbf{r},j)}b), \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $not(aS_{\mathbf{r}}b)$.

Proof. See Appendix A. □

Note 2.4. *Until now, we have applied the concepts of the hierarchy of criteria to one specific ELECTRE method that is the ELECTRE IS method. The Multiple Criteria Hierarchy Process (MCHP) can be applied also to other ELECTRE methods, including the most popular ELECTRE III method. ELECTRE III builds, for each couple of alternatives $(a, b) \in B$, the credibility index*

$$\rho(a,b) = C(a,b) \prod_{\{j:d_j(a,b)>C(a,b)\}} \frac{1-d_j(a,b)}{1-C(a,b)}$$

where for each criterion g_j,

$$d_j(a,b) = \begin{cases} 1, & \textit{if } \; g_j(a) \leq g_j(b) - v_j, \\ \frac{g_j(a)-[g_j(b)-p_j]}{v_j-p_j}, & \textit{if } \; g_j(b) - v_j < g_j(a) < g_j(b) - p_j, \\ 0, & \textit{if } \; g_j(a) \geq g_j(b) - p_j. \end{cases}$$

In MCHP, for each criterion $G_{\mathbf{r}}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, we can define the following credibility index

$$\rho_{\mathbf{r}}(a,b) = C_{\mathbf{r}}(a,b) \prod_{\{\mathbf{t} \in E(G_{\mathbf{r}})\,:\,d_{\mathbf{t}}(a,b)>C_{\mathbf{r}}(a,b)\}} \frac{1-d_{\mathbf{t}}(a,b)}{1-C_{\mathbf{r}}(a,b)}$$

where $d_{\mathbf{t}}(a,b)$ is defined equivalently to $d_j(a,b)$ for each elementary subcriterion $g_{\mathbf{t}}$, $\mathbf{t} \in EL$.
From the definition of $\rho_{\mathbf{r}}(a,b)$ it follows that if none of the elementary subcriteria descending from $G_{\mathbf{r}}$ opposes veto to the outranking of a over b on criterion $G_{\mathbf{r}}$ (that is $d_{\mathbf{r}}(a,b) = 0$ for all $\mathbf{t} \in E(G_{\mathbf{r}})$), then $\rho_{\mathbf{r}}(a,b) = C_{\mathbf{r}}(a,b)$; if some elementary subcriterion descending from $G_{\mathbf{r}}$ opposes the veto to the outranking of a over b with respect to criterion $G_{\mathbf{r}}$ (that is there exists at least one elementary subcriterion $g_{\mathbf{t}}$, with

$\mathbf{t} \in E(G_\mathbf{r})$, *such that* $d_\mathbf{t}(a,b) = 1$*), then* $\rho_\mathbf{r}(a,b) = 0$ *and in all other cases the credibility index* $\rho_\mathbf{r}(a,b)$ *is lower than the concordance index* $C_\mathbf{r}(a,b)$.

3 ROR applied to Hierarchical ELECTRE

3.1 Hierarchical ELECTRE^{GKMS}

The only information the DM can obtain from the evaluations of alternatives with respect to the considered criteria is the dominance relation. In general, information carried by the dominance relation is very poor, and thus, in order to arrive to a final decision which would be concordant with the value system of the DM, it is useful to take into account some preference information provided by the DM. This preference information can be obtained in either direct or indirect way: if the way is direct, then the DM provides precise values or interval of values for the parameters present in the model, and if the way is indirect, then the DM is invited to provide preference information from which the parameters of the model can be inferred.

In this paper, we use a mix of both ways in order to infer the parameters of the model. We suppose that, considering a multiple criteria choice or ranking problem, the DM can provide preference information of two types:

- pairwise comparisons of some reference alternatives from set $A^R \subseteq A$, stating the truth or falsity of outranking relation $aS_\mathbf{r}b$, with $\mathbf{r} \in \mathcal{I}_\mathcal{G} \setminus EL$ and $a, b \in A^R$ (in the following $B^R = A^R \times A^R$),
- information regarding the indifference and preference thresholds $q_\mathbf{t}$ and $p_\mathbf{t}$ for each elementary subcriterion $g_\mathbf{t}, \mathbf{t} \in EL$ and information regarding weights $k_\mathbf{t}$ for some elementary subcriterion $g_\mathbf{t}, \mathbf{t} \in EL$.

Regarding the direct preference information, the DM can provide intervals of possible values $[q_{\mathbf{t},*}, q_\mathbf{t}^*]$ and $[p_{\mathbf{t},*}, p_\mathbf{t}^*]$ for each indifference and preference threshold $q_\mathbf{t}, p_\mathbf{t}$, $\mathbf{t} \in EL$, where $q_{\mathbf{t},*}$ and $q_\mathbf{t}^*$ are, respectively, the smallest and the greatest value of the indifference threshold, and $p_{\mathbf{t},*}$ and $p_\mathbf{t}^*$ are, respectively, the smallest and the greatest value of the preference threshold allowed by the DM.

Besides, we assume that the DM could give information on the weight $k_\mathbf{t}$ of some elementary subcriterion providing interval of possible values $[k_{\mathbf{t},*}, k_\mathbf{t}^*]$, where $k_{\mathbf{t},*}$ and $k_\mathbf{t}^*$ are, respectively, the smallest and the greatest value of the weights allowed by the DM, or providing pairwise comparison between the elementary subcriteria of the type: "elementary subcriterion $g_{\mathbf{t_1}}$ is more important than elemenatry subcriterion $g_{\mathbf{t_2}}$" or "elementary subcriteria $g_{\mathbf{t_1}}$ and $g_{\mathbf{t_2}}$ are equally important" that are translated from the constraints $k_{\mathbf{t_1}} > k_{\mathbf{t_2}}$ and $k_{\mathbf{t_1}} = k_{\mathbf{t_2}}$ respectively.

If the DM cannot provide intervals of indifference and preference threshold values for an elementary subcriterion $g_\mathbf{t}$, then (s)he has to indicate at least one couple of reference alternatives $a, b \in A^R \subseteq A$ for which the difference between $g_{\mathbf{t}}(a)$ and $g_{\mathbf{t}}(b)$ is non-significant for the DM ($a \sim_{\mathbf{t}} b$), and at least one couple of reference alternatives a, b for which the difference between $g_{\mathbf{t}}(a)$ and $g_{\mathbf{t}}(b)$ is significant for the DM ($a \succ_{\mathbf{t}} b$). We denote by EL_1 and EL_2 the subsets of EL (such that $EL_1 \cup EL_2 = EL$) containing indices of elementary subcriteria for which the DM provides information about the thresholds in a direct or indirect way, respectively.

In order to ensure the consistency of the above thresholds, the following constraints need to be satisfied:

- $q_{\mathbf{t},*} \leq q^*_{\mathbf{t}}$, $p_{\mathbf{t},*} \leq p^*_{\mathbf{t}}$ and $q^*_{\mathbf{t}} \leq p_{\mathbf{t},*}$, for all $\mathbf{t} \in EL_1$,
- $|g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b)| < g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d)$, if $a \sim_{\mathbf{t}} b$ and $c \succ_{\mathbf{t}} d$, for all $\mathbf{t} \in EL_2$,
- $p^*_{\mathbf{t}}$ should be not greater than $\beta_{\mathbf{t}} - \alpha_{\mathbf{t}}, \mathbf{t} \in EL_1$, where $\alpha_{\mathbf{t}} = \min_{a \in A} g_{\mathbf{t}}(a)$, and $\beta_{\mathbf{t}} = \max_{a \in A} g_{\mathbf{t}}(a)$.

We call *compatible model*, a set of parameters (thus variables $\psi_{\mathbf{t}}(a,b)$ for each pair of alternatives $(a,b) \in B$ and for each elementary subcriterion $g_{\mathbf{t}}$, $\mathbf{t} \in EL$, veto thresholds $v_{\mathbf{t}}$ for all $\mathbf{t} \in EL$, and concordance cutting levels $\lambda_{\mathbf{s}}$ for all $\mathbf{s} \in LBO$) which restore the preference information provided by the DM and thus satisfy the following set of constraints (see [9] for a similar formulation in a non-hierarchical case, and Appendix B for a detailed description of the constraints):

$$
\left.
\begin{array}{l}
\text{Pairwise comparison stating } aS_{\mathbf{r}}b \text{ or } not(aS_{\mathbf{r}}b)\text{:}\\[4pt]
\quad C_{\mathbf{r}}(a,b)=\sum_{\mathbf{t}\in E(G_{\mathbf{r}})}\psi_{\mathbf{t}}(a,b)\geq\lambda_{\mathbf{r}} \;\text{ and }\; g_{\mathbf{t}}(b)-g_{\mathbf{t}}(a)+\varepsilon\leq v_{\mathbf{t}},\; \mathbf{t}\in E(G_{\mathbf{r}}),\\
\qquad \text{if } aS_{\mathbf{r}}b, \;\; \text{for all } (a,b)\in B^R,\\[4pt]
\quad C_{\mathbf{r}}(a,b)=\sum_{\mathbf{t}\in E(G_{\mathbf{r}})}\psi_{\mathbf{t}}(a,b)+\varepsilon\leq\lambda_{\mathbf{r}}+M_0^{\mathbf{r}}(a,b) \;\text{ and }\; g_{\mathbf{t}}(b)-g_{\mathbf{t}}(a)\geq v_{\mathbf{t}}-\delta_{\mathbf{r}}M_{\mathbf{t}}(a,b),\\
\qquad \text{if } not(aS_{\mathbf{r}}b), \;\; \text{for } \;(a,b)\in B^R,\\
\qquad M_0^{\mathbf{r}}(a,b), M_{\mathbf{t}}(a,b)\in\{0,1\}, \text{ for all } \mathbf{t}\in E(G_{\mathbf{r}}),\; M_0^{\mathbf{r}}(a,b)+\sum_{\mathbf{t}\in E(G_{\mathbf{r}})}M_{\mathbf{t}}(a,b)\leq|E(G_{\mathbf{r}})|\,,\\[4pt]
\qquad \delta_{\mathbf{r}}\geq max_{\mathbf{t}\in E(G_{\mathbf{r}})}\{\beta_{\mathbf{t}}-\alpha_{\mathbf{t}}\} \text{ where } \alpha_{\mathbf{t}}=\min_{a\in A}g_{\mathbf{t}}(a) \text{ and } \beta_{\mathbf{t}}=\max_{a\in A}g_{\mathbf{t}}(a).\\[4pt]
\text{Concordance cutting levels and values of inter-criteria parameters:}\\[4pt]
\quad \lambda_{\mathbf{s}}\geq\sum_{\mathbf{t}\in E(G_{\mathbf{s}})}\dfrac{\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*})}{2}, \;\text{ and } \lambda_{\mathbf{s}}\leq\sum_{\mathbf{t}\in E(G_{\mathbf{s}})}\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*}), \text{ for all } \mathbf{s}\in LBO,\\
\quad \lambda_{\mathbf{r}}=\sum_{j=1}^{n(\mathbf{r})}\lambda_{(\mathbf{r},j)}, \text{ for all } \mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus\{LBO\cup EL\}\,,\\
\quad \sum_{\mathbf{t}\in EL}\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*})=1, \text{ where } x_{\mathbf{t},*},x_{\mathbf{t}}^*\in A \text{ for all } \mathbf{t}\in EL: g_{\mathbf{t}}(x_{\mathbf{t}}^*)=\beta_{\mathbf{t}}, \text{ and } g_{\mathbf{t}}(x_{\mathbf{t},*})=\alpha_{\mathbf{t}},\\
\quad v_{\mathbf{t}}\geq p_{\mathbf{t}}^*+\varepsilon,\; \mathbf{t}\in EL,\\[4pt]
\quad v_{\mathbf{t}}\geq g_{\mathbf{t}}(b)-g_{\mathbf{t}}(a)+\varepsilon \;\text{ if }\; a\sim_{\mathbf{t}}b, \text{ and } g_{\mathbf{t}}(a)\leq g_{\mathbf{t}}(b),\; \mathbf{t}\in EL_2, \text{ for all } (a,b)\in B,\\[4pt]
\text{Values of marginal concordance indices conditioned by intra-criterion preference information,}\\
\text{for all } (a,b)\in B\text{:}\\[4pt]
\quad k_{\mathbf{t},*}\leq\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*})\leq k_{\mathbf{t}}^*, \mathbf{t}\in EL,\\[4pt]
\quad \psi_{\mathbf{t_1}}(x_{\mathbf{t_1}}^*,x_{\mathbf{t_1},*})\geq\psi_{\mathbf{t_2}}(x_{\mathbf{t_2}}^*,x_{\mathbf{t_2},*})+\varepsilon, \;\; \text{if elementary subcriterion } g_{\mathbf{t_1}} \text{ is more important than}\\
\qquad \text{elementary subcriterion } g_{\mathbf{t_2}}, \mathbf{t_1},\mathbf{t_2}\in EL,\\[4pt]
\quad \psi_{\mathbf{t_1}}(x_{\mathbf{t_1}}^*,x_{\mathbf{t_1},*})=\psi_{\mathbf{t_2}}(x_{\mathbf{t_2}}^*,x_{\mathbf{t_2},*}), \;\; \text{if elementary subcriteria } g_{\mathbf{t_1}} \text{ and } g_{\mathbf{t_2}} \text{ are}\\
\qquad \text{equally important}, \mathbf{t_1},\mathbf{t_2}\in EL,\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)=0 \;\text{ if }\; g_{\mathbf{t}}(a)-g_{\mathbf{t}}(b)\leq -p_{\mathbf{t}}^*, \mathbf{t}\in EL_1,\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)\geq\varepsilon \;\text{ if }\; g_{\mathbf{t}}(a)-g_{\mathbf{t}}(b)> -p_{\mathbf{t},*}, \mathbf{t}\in EL_1,\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)=\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*}) \;\text{ if }\; g_{\mathbf{t}}(a)-g_{\mathbf{t}}(b)\geq -q_{\mathbf{t},*}, \mathbf{t}\in EL_1,\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)+\varepsilon\leq\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*}) \;\text{ if }\; g_{\mathbf{t}}(a)-g_{\mathbf{t}}(b)< -q_{\mathbf{t}}^*, \mathbf{t}\in EL_1,\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)=\psi_{\mathbf{t}}(b,a)=\psi_{\mathbf{t}}(x_{\mathbf{t}}^*,x_{\mathbf{t},*}) \;\text{ if }\; a\sim_{\mathbf{t}}b, \mathbf{t}\in EL_2\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)=0 \;\text{ if }\; b\succ_{\mathbf{t}}a, \mathbf{t}\in EL_2.\\[4pt]
\text{Monotonicity of the functions of marginal concordance indices, for all } a,b,c,d\in A,\; \mathbf{t}\in EL\text{:}\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)\geq\psi_{\mathbf{t}}(c,d) \;\text{ if }\; g_{\mathbf{t}}(a)-g_{\mathbf{t}}(b)>g_{\mathbf{t}}(c)-g_{\mathbf{t}}(d),\\[4pt]
\quad \psi_{\mathbf{t}}(a,b)=\psi_{\mathbf{t}}(c,d) \;\text{ if }\; g_{\mathbf{t}}(a)-g_{\mathbf{t}}(b)=g_{\mathbf{t}}(c)-g_{\mathbf{t}}(d),
\end{array}
\right\} E^{A^R}
$$

The whole set of constraints E^{A^R} has the form of 0-1 Mixed Integer Linear Program (MILP), as shown above. If E^{A^R} is feasible and $\varepsilon^* = max\ \varepsilon$, subject to E^{A^R}, is greater than 0, then there exists at least one outranking model compatible with the preference information.

In general, there may exist more than one outranking model compatible with preference information provided by the DM; each one of the compatible models restores the preference information concerning the reference alternatives provided by the DM, but it can compare in a different way the other couples of alternatives not present in the preference information provided by the DM. For this reason, ROR takes into account all outranking models compatible with preference information provided by the DM simultaneously. In the ROR context, in case of the hierarchy of criteria applied to ELECTRE, and considering a criterion/subcriterion $G_{\mathbf{r}}$ of the hierarchy, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ and two alternatives $a, b \in A$, we can give the following definitions:

Definition 3.1.

- *a necessarily outranks b with respect to $G_{\mathbf{r}}$, and we write $aS^N_{\mathbf{r}}b$, if a outranks b with respect to $G_{\mathbf{r}}$, for all compatible models,*
- *a possibly outranks b with respect to $G_{\mathbf{r}}$, and we write $aS^P_{\mathbf{r}}b$, if a outranks b with respect to $G_{\mathbf{r}}$, for at least one compatible model,*
- *a necessarily does not outrank b with respect to $G_{\mathbf{r}}$, and we write $aS^{CN}_{\mathbf{r}}b$, if a does not outrank b with respect to $G_{\mathbf{r}}$, for all compatible models,*
- *a possibly does not outrank b with respect to $G_{\mathbf{r}}$, and we write $aS^{CP}_{\mathbf{r}}b$, if a does not outrank b with respect to $G_{\mathbf{r}}$, for at least one compatible model.*

Remark that, in case of $\mathbf{r} = 0$, the necessary and possible outranking relations $S^N_{\mathbf{r}}$ and $S^P_{\mathbf{r}}$ are the same as necessary and possible outranking relations defined in [9], for a flat (non-hierarchical) structure of the set of criteria.

Given a pair of alternatives $(a, b) \in B$, and a criterion $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, necessary and possible outranking relations ($\succsim^N_{\mathbf{r}}$, $\succsim^P_{\mathbf{r}}$) can be computed as follows.

- To check whether $aS^N_{\mathbf{r}}b$, we assume that a does not outrank b with respect to criterion $G_{\mathbf{r}}$ ($not(aS_{\mathbf{r}}b)$), and we add the corresponding constraints to set E^{A^R}, getting the set of constraints $E^N_{\mathbf{r}}(a, b)$ shown below. Then, we verify whether $not(aS_{\mathbf{r}}b)$ is possible in the set of all outranking models compatible with the previously provided preference information.

$$\left.\begin{array}{l} E^{A^R} \\ C_{\mathbf{r}}(a,b) = \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} \psi_{\mathbf{t}}(a,b) + \varepsilon \leq \lambda_{\mathbf{r}} + M_0^{\mathbf{r}}(a,b) \;\text{ and }\; g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq v_{\mathbf{t}} - \delta_{\mathbf{r}} M_{\mathbf{t}}(a,b), \\ M_0^{\mathbf{r}}(a,b) + \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} M_{\mathbf{t}}(a,b) \leq |E(G_{\mathbf{r}})|,\; M_0^{\mathbf{r}}(a,b), M_{\mathbf{t}}(a,b) \in \{0,1\}, \;\; \mathbf{t} \in E(G_{\mathbf{r}}). \end{array}\right\} E_{\mathbf{r}}^N(a,b)$$

We say that:

$aS_{\mathbf{r}}^N b$ if $E_{\mathbf{r}}^N(a,b)$ is infeasible or $\varepsilon_{\mathbf{r}}^N(a,b) \leq 0$ where $\varepsilon_{\mathbf{r}}^N(a,b) = max\ \varepsilon$, subject to $E_{\mathbf{r}}^N(a,b)$.

- To check whether $aS_{\mathbf{r}}^P b$, we assume that a outranks b with respect to criterion $G_{\mathbf{r}}$ ($aS_{\mathbf{r}}b$), and we add the corresponding constraints to the set E^{A^R}, getting the set of constraints $E_{\mathbf{r}}^P(a,b)$ shown below. Then, we verify whether $aS_{\mathbf{r}}b$ is possible in the set of all outranking models compatible with the previously provided preference information.

$$\left.\begin{array}{l} E^{A^R} \\ C_{\mathbf{r}}(a,b) = \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} \psi_{\mathbf{t}}(a,b) \geq \lambda_{\mathbf{r}} \;\text{ and }\; g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) + \varepsilon \leq v_{\mathbf{t}},\; \mathbf{t} \in E(G_{\mathbf{r}}) \end{array}\right\} E_{\mathbf{r}}^P(a,b)$$

We say that:

$aS_{\mathbf{r}}^P b$ if $E_{\mathbf{r}}^P(a,b)$ is feasible and $\varepsilon_{\mathbf{r}}^P(a,b) > 0$ where $\varepsilon_{\mathbf{r}}^P(a,b) = max\ \varepsilon$, subject to $E_{\mathbf{r}}^P(a,b)$.

Note 3.1. *It is worth noting that the set of constraints E^{A^R} defines a set of variables $\psi_{\mathbf{t}}(a,b)$, $\mathbf{t} \in EL$, $(a,b) \in B$, being non-decreasing functions with respect to the difference $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b)$, differently from the functions $\phi_{\mathbf{t}}(a,b)$ which are non-decreasing and linear. This is due to the fact that indifference and preference thresholds, as well as the veto thresholds, are not directly provided by the DM. In this situation, taking thresholds and weights as unknown variables, makes that the optimization problems to be solved in ROR are no more linear programming ones. As there are many optimization problems to be solved in ROR, the whole approach would be practically non-tractable.*
If the DM would be able to provide all the thresholds considered in the model (indifference, preference and veto), then linear programming could be applied again within a simplified model of ROR, considering as variables only the weights $k_{\mathbf{t}}, \mathbf{t} \in EL$, and the concordance cutting levels $\lambda_{\mathbf{s}}$, $\mathbf{s} \in LBO$. In this case, the feasibility constraints of the optimization problems considered in ROR can be simply modified to the following form:

$$\left.\begin{array}{l}
\textit{Pairwise comparison stating } aS_{\mathbf{r}}b \textit{ or not}(aS_{\mathbf{r}}b)\textit{:}\\
\quad C_{\mathbf{r}}(a,b) = \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} k_{\mathbf{t}} \cdot \phi_{\mathbf{t}}(a,b) \geq \lambda_{\mathbf{r}}, \quad \textit{if} \;\; aS_{\mathbf{r}}b, \quad \textit{for} \;\; (a,b)\in B^R,\\
\quad C_{\mathbf{r}}(a,b) = \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} k_{\mathbf{t}} \cdot \phi_{\mathbf{t}}(a,b) + \varepsilon \leq \lambda_{\mathbf{r}} \;\; \textit{if} \;\; \textit{not}(aS_{\mathbf{r}}b), \quad \textit{for} \;\; (a,b)\in B^R,\\
\textit{Concordance cutting levels and values of inter-criteria parameters:}\\
\quad \lambda_{\mathbf{s}} \geq \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{s}})} \frac{k_{\mathbf{t}}}{2}, \;\; \textit{and}\, \lambda_{\mathbf{s}} \leq \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{s}})} k_{\mathbf{t}}, \; \textit{for all}\, \mathbf{s}\in LBO,\\
\quad \displaystyle\sum_{\mathbf{t}\in EL} k_{\mathbf{t}} = 1,\\
\quad k_{\mathbf{t},*} \leq k_{\mathbf{t}} \leq k_{\mathbf{t}}^{*}, \mathbf{t}\in EL,\\
\quad k_{\mathbf{t_1}} \geq k_{\mathbf{t_2}} + \varepsilon, \quad \textit{if elementary subcriterion } g_{\mathbf{t_1}} \textit{ is more important than}\\
\qquad\quad \textit{elementary subcriterion } g_{\mathbf{t_2}}, \mathbf{t_1}, \mathbf{t_2} \in EL,\\
\quad k_{\mathbf{t_1}} = k_{\mathbf{t_2}}, \quad \textit{if elementary subcriteria } g_{\mathbf{t_1}} \textit{ and } g_{\mathbf{t_2}} \textit{ are}\\
\qquad\quad \textit{equally important}, \mathbf{t_1}, \mathbf{t_2} \in EL,
\end{array}\right\} E^{A^R}$$

$$\left.\begin{array}{l}
E^{A^R}\\
C_{\mathbf{r}}(a,b) = \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} k_{\mathbf{t}} \cdot \phi_{\mathbf{t}}(a,b) + \varepsilon \leq \lambda_{\mathbf{r}},
\end{array}\right\} E^N_{\mathbf{r}}(a,b), \qquad
\left.\begin{array}{l}
E^{A^R}\\
C_{\mathbf{r}}(a,b) = \displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})} k_{\mathbf{t}} \cdot \phi_{\mathbf{t}}(a,b) \geq \lambda_{\mathbf{r}},
\end{array}\right\} E^P_{\mathbf{r}}(a,b)$$

The linearity and simplicity of the above formulation is concordant with reasoning of Note 2.1.

Remark that the preference information of the type $aS_{\mathbf{r}}b$ *and* $not(aS_{\mathbf{r}}b)$ *provided by the DM involves the concordance test only, because the veto thresholds were given before by the DM, and thus not* $(aS_{\mathbf{r}}b)$ *could not reasonably be caused by discordance. Indeed, it is reasonable to assume that the DM stating that* $aS_{\mathbf{r}}b$ *or* $not(aS_{\mathbf{r}}b)$ *already knows that* $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) < v_{\mathbf{t}}$ *for all* $\mathbf{t} \in E(G_{\mathbf{r}})$.

3.2 Properties of necessary and possible outranking relations

Proposition 3.1.

1. *For all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $S^N_{\mathbf{r}} \subseteq S^P_{\mathbf{r}}$,
2. *For all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $S^P_{\mathbf{r}}$ *and* $S^N_{\mathbf{r}}$ *are reflexive,*
3. *For all* $a, b \in A$, *for all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $aS^N_{\mathbf{r}}b \Leftrightarrow not(aS^{CP}_{\mathbf{r}}b)$,

4. *For all* $a, b \in A$, *for all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $aS^{P}_{\mathbf{r}}b \Leftrightarrow not(aS^{CN}_{\mathbf{r}}b)$,

5. $S^{CN}_{\mathbf{r}} \subseteq S^{CP}_{\mathbf{r}}$, *for all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$,

6. *For all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $S^{CP}_{\mathbf{r}}$ *and* $S^{CN}_{\mathbf{r}}$ *are irreflexive.*

Proof. See Appendix A. □

Proposition 3.2.

1. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (EL \cup LBO)$, *such that*

$$aS^{N}_{(\mathbf{r},j)}b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aS^{N}_{\mathbf{r}}b$,

2. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (EL \cup LBO)$, *such that:*

   α) $aS^{N}_{(\mathbf{r},j)}b$ *for all* $j = 1, \ldots, n(\mathbf{r}), j \neq w$,

   β) $aS^{P}_{(\mathbf{r},w)}b$,

   then $aS^{P}_{\mathbf{r}}b$,

3. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (EL \cup LBO)$, *such that*

$$aS^{CN}_{(\mathbf{r},j)}b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aS^{CN}_{\mathbf{r}}b$.

4. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (EL \cup LBO)$, *such that:*

   α) $aS^{CN}_{(\mathbf{r},j)}b$ *for all* $j = 1, \ldots, n(\mathbf{r}), j \neq w$,

   β) $aS^{CP}_{(\mathbf{r},w)}b$,

   then $aS^{CP}_{\mathbf{r}}b$.

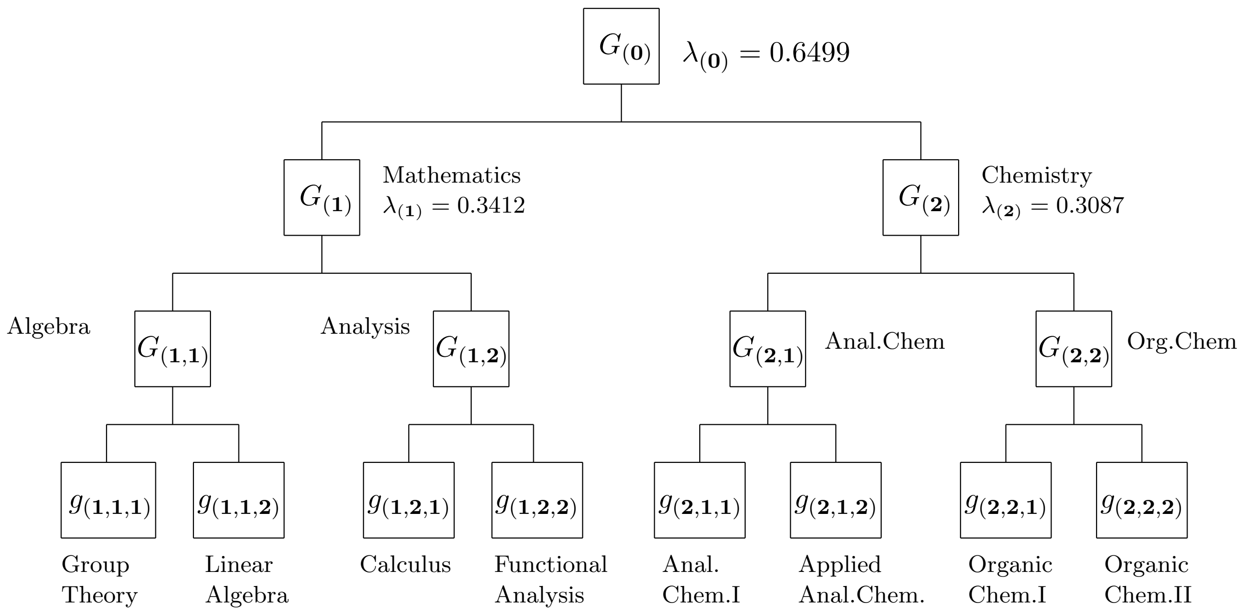
Proof. See Appendix A. □

3.3 An illustrative example

In this section, we present an illustrative example in order to show how to use the ELECTRE method and the ELECTRE^{GKMS} method in case of the hierarchical structure of criteria. At first, we describe how to use ELECTRE method in case of availability of full preference information composed of the weights, the preference, indifference and veto thresholds, and about the concordance cutting levels.

Let us suppose that a university department of natural sciences, like every year, has a possibility of funding a scholarship for one of its best students. As five best students passed to the final selection, the Dean has to choose from among them one laureate. These five finalists are evaluated with respect to two macro-subjects: Mathematics and Chemistry. Both these macro-subjects present a hierarchical structure; on one hand, Mathematics has two sub-subjects: Algebra and Analysis, and each one of these has other two sub-subjects: Group Theory and Linear Algebra are sub-subjects of Algebra, while Calculus and Functional Analysis are sub-subjects of Analysis. On the other hand, Chemistry has two sub-subjects: Analytical Chemistry and Organic Chemistry, and each one of them has two sub-subjects: Analytical Chemistry I and Applied Analytical Chemistry are sub-subjects of Analytical Chemistry, while Organic Chemistry I and Organic Chemistry II are sub-subjects of Organic Chemistry. The described hierarchy of criteria is shown in Figure 5.

Figure 5: Hierarchical evaluation of students



The eight sub-subjects are thus the elementary subcriteria of the considered hierarchical structure and the evaluations of the students with respect to the eight sub-subjects are shown in Table 1. The evaluation of students on these elementary subcriteria is included between 18 and 30. Weights, indifference, preference and veto thresholds are shown in Table 2(a).

Table 1: Evaluations of students on elementary subcriteria

Student	$g_{(\mathbf{1},\mathbf{1},\mathbf{1})}$	$g_{(\mathbf{1},\mathbf{1},\mathbf{2})}$	$g_{(\mathbf{1},\mathbf{2},\mathbf{1})}$	$g_{(\mathbf{1},\mathbf{2},\mathbf{2})}$	$g_{(\mathbf{2},\mathbf{1},\mathbf{1})}$	$g_{(\mathbf{2},\mathbf{1},\mathbf{2})}$	$g_{(\mathbf{2},\mathbf{2},\mathbf{1})}$	$g_{(\mathbf{2},\mathbf{2},\mathbf{2})}$
s_1	28	22	27	21	29	21	28	20
s_2	20	23	19	22	30	20	29	19
s_3	29	21	28	20	18	24	18	23
s_4	30	20	29	19	28	22	27	21
s_5	18	24	18	23	20	23	19	22

The Dean decides to provide information regarding the concordance cutting levels in the way explained in the Note 2.3. Then, (s)he states that each concordance cutting level $\lambda_{\mathbf{s}}$, $\mathbf{s} \in LBO$, should be equal to 65% of the sum of the weights of elementary subcriteria descending from $G_{\mathbf{s}}$. In consequence, for each criterion of the last but one level the values of concordance cutting levels are: $\lambda_{(\mathbf{1},\mathbf{1})} = 0.1625$, $\lambda_{(\mathbf{1},\mathbf{2})} = 0.1787$, $\lambda_{(\mathbf{2},\mathbf{1})} = 0.1462$ and $\lambda_{(\mathbf{2},\mathbf{2})} = 0.1625$ as reported in Table 2(b).

Table 2: ELECTRE parameters in case of the hierarchy of criteria

(a) Weights and thresholds

Elementary subcriterion, $g_{\mathbf{t}}$	$k_{\mathbf{t}}$	$q_{\mathbf{t}}$	$p_{\mathbf{t}}$	$v_{\mathbf{t}}$
Group Theory	0.1	1	4	10
Linear Algebra	0.15	1	4	10
Calculus	0.125	1	4	10
Functional Analysis	0.15	1	4	10
Analytical Chemistry I	0.1	2	5	10
App. Anal. Chemistry	0.125	2	5	10
Organic Chemistry I	0.15	2	5	10
Organic Chemistry II	0.1	2	5	10

(b) Concordance cutting levels

Criterion, $G_{\mathbf{r}}$	$\lambda_{\mathbf{r}}$
Algebra	0.1625
Analysis	0.1787
Analytical Chemistry	0.1462
Organic Chemistry	0.1625

Following the procedure explained in section 2.2, we obtain the outranking relations shown in Table 3, where:

$$S_{(\mathbf{r})}(s_1, s_2) = \begin{cases} 1 & \text{if} \quad s_1 \text{ outranks } s_2 \text{ with respect to criterion } G_{\mathbf{r}}, \\ 0 & \text{if} \quad s_1 \text{ does not outrank } s_2 \text{ with respect to criterion } G_{\mathbf{r}}. \end{cases}$$

In Table 3, we obtain the "overall outranking relation", that is the outranking relation with respect to the totality of criteria, as well as "partial outranking relations", that is outranking relations with respect to a particular subcriterion at a given level of the hierarchy.

Remark that using the classical ELECTRE method, we obtain the overall outranking relation only because all criteria are considered at the same level. In consequence, using the classical ELECTRE method, we could learn that student s_2 does not outrank student s_4 with respect to the totality of criteria, but we could

Table 3: Outranking relations at particular levels of the hierarchy of criteria

$S_{(\mathbf{0})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	1	1	0	0	1
s_3	0	0	1	0	1
s_4	1	1	1	1	1
s_5	0	0	0	0	1

$S_{(\mathbf{1})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	0	1	0	0	1
s_3	1	1	1	1	0
s_4	1	0	1	1	0
s_5	0	1	0	0	1

$S_{(\mathbf{2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	1	1	1	1	1
s_3	0	0	1	0	1
s_4	1	1	1	1	1
s_5	0	0	1	0	1

$S_{(\mathbf{1,1})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	0	1	0	0	1
s_3	1	1	1	1	0
s_4	1	0	1	1	0
s_5	0	1	0	0	1

$S_{(\mathbf{1,2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	0	1	0	0	1
s_3	1	1	1	1	0
s_4	1	0	1	1	0
s_5	0	1	0	0	1

$S_{(\mathbf{2,1})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	1	1	0	1	1
s_3	0	0	1	0	1
s_4	1	1	1	1	1
s_5	0	0	1	0	1

$(\mathbf{2,2})$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	1	1
s_2	1	1	1	1	1
s_3	0	0	1	0	1
s_4	1	1	1	1	1
s_5	0	0	1	0	1

not know that student s_2 outranks student s_4 with respect to Chemistry, Analytical Chemistry and Organic Chemistry.

According to point 1 of Proposition 2.1, we observe that if student s_1 outranks student s_5 with respect to Mathematics ($G_{(\mathbf{1})}$) and Chemistry ($G_{(\mathbf{2})}$), then s_1 outranks s_5 with respect to the totality of criteria ($G_{(\mathbf{0})}$), but the contrary is not true; in fact, for example, student s_2 outranks student s_3 with respect to Chemistry, but at the same time student s_2 does not outrank student s_3 with respect to Analytical Chemistry ($G_{(\mathbf{2,1})}$), being a sub-criterion descending from Chemistry.

Now, let us suppose that the Dean cannot provide the full preference information regarding the parameters of the Hierarchical ELECTRE method. The only information the Dean can get from the evaluation table is the dominance relation, but in this particular case there is no student dominating another student. Thus, the Dean decides to use the Hierarchical ELECTRE^{GKMS} method. In fact, (s)he realizes that using this procedure, (s)he has two advantages: (s)he can give finer preference information, taking into account subsets of criteria at different levels of the hierarchy, and at the same time, (s)he can get more information from the partial necessary and possible outranking relations. In order to use this methodology, (s)he provides the thresholds shown in Table 4.

Looking at the evaluations of students shown in Table 1, the Dean specifies the following pairwise comparisons:

- student s_4 outranks student s_2 with respect to Mathematics ($s_4 S_{(\mathbf{1})} s_2$),

Table 4: Indifference and preference thresholds provided by the Dean

Elementary subcriterion, $g_{\mathbf{t}}$	$q_{\mathbf{t},*}$	$q^*_{\mathbf{t}}$	$p_{\mathbf{t},*}$	$p^*_{\mathbf{t}}$
Group Theory	1	2	3	4
Linear Algebra	1	2	3	4
Calculus	1	2	3	4
Functions Theory	1	2	3	4
Analytical Chemistry I	1	2	3	4
App. Anal. Chemistry	1	2	3	4
Organic Chemistry I	1	2	3	4
Organic Chemistry II	1	2	3	4

- student s_5 does not outrank student s_1 with respect to Organic Chemistry ($not(s_5 S_{(\mathbf{2},\mathbf{2})} s_1)$).

These two pieces of information, are translated into the following constraints regarding variables of the ordinal regression problem:

- $s_4 S_{(\mathbf{1})} s_2$ is translated into:

  1. $\psi_{(\mathbf{1},\mathbf{1},\mathbf{1})}(s_4, s_1) + \psi_{(\mathbf{1},\mathbf{1},\mathbf{2})}(s_4, s_1) + \psi_{(\mathbf{1},\mathbf{2},\mathbf{1})}(s_4, s_1) + \psi_{(\mathbf{1},\mathbf{2},\mathbf{2})}(s_4, s_1) \geq \lambda_{(\mathbf{1},\mathbf{1})} + \lambda_{(\mathbf{1},\mathbf{2})}$,
  2. $v_{(\mathbf{1},\mathbf{1},\mathbf{1})} \geq g_{(\mathbf{1},\mathbf{1},\mathbf{1})}(s_2) - g_{(\mathbf{1},\mathbf{1},\mathbf{1})}(s_4) + \varepsilon = -10 + \varepsilon$,
  3. $v_{(\mathbf{1},\mathbf{1},\mathbf{2})} \geq g_{(\mathbf{1},\mathbf{1},\mathbf{2})}(s_2) - g_{(\mathbf{1},\mathbf{1},\mathbf{2})}(s_4) + \varepsilon = 3 + \varepsilon$,
  4. $v_{(\mathbf{1},\mathbf{2},\mathbf{1})} \geq g_{(\mathbf{1},\mathbf{2},\mathbf{1})}(s_2) - g_{(\mathbf{1},\mathbf{2},\mathbf{1})}(s_4) + \varepsilon = -10 + \varepsilon$,
  5. $v_{(\mathbf{1},\mathbf{2},\mathbf{2})} \geq g_{(\mathbf{1},\mathbf{2},\mathbf{2})}(s_2) - g_{(\mathbf{1},\mathbf{2},\mathbf{2})}(s_4) + \varepsilon = 3 + \varepsilon$.

- $not(s_5 S_{(\mathbf{2},\mathbf{2})} s_1)$ is translated into:

  1. $\psi_{(\mathbf{2},\mathbf{2},\mathbf{1})}(s_5, s_1) + \psi_{(\mathbf{2},\mathbf{2},\mathbf{2})}(s_5, s_1) + \varepsilon \leq \lambda_{(\mathbf{2},\mathbf{2})} + M_0^{(\mathbf{2},\mathbf{2})}(s_5, s_1)$,
  2. $v_{(\mathbf{2},\mathbf{2},\mathbf{1})} - \delta M_{(\mathbf{2},\mathbf{2},\mathbf{1})}(5, 1) \leq g_{(\mathbf{2},\mathbf{2},\mathbf{1})}(s_1) - g_{(\mathbf{2},\mathbf{2},\mathbf{1})}(s_5) = 9$,
  3. $v_{(\mathbf{2},\mathbf{2},\mathbf{2})} - \delta M_{(\mathbf{2},\mathbf{2},\mathbf{2})}(5, 1) \leq g_{(\mathbf{2},\mathbf{2},\mathbf{2})}(s_1) - g_{(\mathbf{2},\mathbf{2},\mathbf{2})}(s_5) = -2$,
  4. $M_0^{(\mathbf{2},\mathbf{2})}(s_5, s_1) + M_{(\mathbf{2},\mathbf{2},\mathbf{1})}(s_5, s_1) + M_{(\mathbf{2},\mathbf{2},\mathbf{2})}(s_5, s_1) \leq 2$,
  5. $M_0^{(\mathbf{2},\mathbf{2})}(s_5, s_1), M_{(\mathbf{2},\mathbf{2},\mathbf{1})}(s_5, s_1), M_{(\mathbf{2},\mathbf{2},\mathbf{2})}(s_5, s_1) \in \{0, 1\}$.

The necessary outranking relation resulting from application of all sets of preference model parameters compatible with the given preference information on the set of five students is presented in Table 5.

Looking at Table 5, we can observe that with respect to the totality of criteria, the only information the Dean obtains is that student s_1 necessarily outranks student s_2 and student s_4 necessarily outranks student s_1. But, when looking at the subcriteria of the hierarchy, the Dean could observe some facts which cannot be seen when using the classic ELECTRE^{GKMS} designed for a flat structure of criteria.

Table 5: Necessary outranking relation obtained from application of the hierarchical version of ELECTRE^{GKMS} (1 means true, and 0 means false)

$S^N_{(\mathbf{0})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	0	0	0
s_2	0	1	0	0	0
s_3	0	0	1	0	0
s_4	1	0	0	1	0
s_5	0	0	0	0	1

$S^N_{(\mathbf{1})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	0	1
s_2	0	1	0	0	1
s_3	1	1	1	1	1
s_4	1	1	1	1	0
s_5	0	0	0	0	1

$S^N_{(\mathbf{2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	0	1	0
s_2	1	1	0	0	0
s_3	0	0	1	0	0
s_4	1	0	0	1	1
s_5	0	0	1	0	1

$S^N_{(\mathbf{1,1})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	0	0
s_2	0	1	0	0	1
s_3	1	0	1	1	0
s_4	0	0	1	1	0
s_5	0	0	0	0	1

$S^N_{(\mathbf{1,2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	1	0	0
s_2	0	1	0	0	1
s_3	1	0	1	1	0
s_4	0	0	1	1	0
s_5	0	1	0	0	1

$S^N_{(\mathbf{2,1})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	0	1	0
s_2	1	1	0	0	0
s_3	0	0	1	0	0
s_4	1	0	0	1	1
s_5	0	0	1	0	1

$S^N_{(\mathbf{2,2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	1	0	1	0
s_2	1	1	0	0	0
s_3	0	0	1	0	1
s_4	1	0	0	1	1
s_5	0	0	1	0	1

According to Proposition 3.2, from the necessary outranking of student s_4 over student s_1 with respect to Mathematics and Chemistry, follows the necessary outranking of student s_4 over student s_1 with respect to the totality of criteria, but at the same time, while student s_4 necessarily outranks student s_1 with respect to Mathematics, student s_4 does not necessarily outrank student s_1 with respect to Algebra being a subcriterion of Mathematics at the level immediately below.

4 Handling the hierarchy of criteria in PROMETHEE methods

In this section, we describe the extension of another outranking method, called PROMETHEE, to the hierarchy of criteria (for a detailed description of PROMETHEE methods in case of a flat structure of criteria see [2]).

In the case of the hierarchy of criteria, PROMETHEE methods compare couples of alternatives with respect to criteria and subcriteria of the hierarchical family of criteria in order to construct an outranking relation in the set of alternatives. This construction involves a few parameters, that is, the weights of elementary subcriteria, as well as indifference and preference thresholds for differences of evaluations of couples of alternatives on each elementary subcriterion. The preference of the DM regarding a couple of alternatives (a, b) with respect to elementary subcriterion $g_{\mathbf{t}}$ depends on the difference between $g_{\mathbf{t}}(a)$ and $g_{\mathbf{t}}(b)$ and for this reason the preference of a over b can be represented by a function $P_{\mathbf{t}}(a, b)$, increasing with $d_{\mathbf{t}}(a, b) = g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b)$. In [2], there are given six different types of functions $P_{\mathbf{t}}(a, b)$, and each one of them involves from zero to three parameters. Let us suppose, there are m evaluation criteria and n alternatives in set A. After the DM has decided which function $P_{\mathbf{t}}$ is expressing the best her/his preferences with respect to elementary subcriterion $g_{\mathbf{t}}$, and after introducing the weights $k_{\mathbf{t}}$ for each elementary subcriterion $g_{\mathbf{t}}, \mathbf{t} \in EL$, one can calculate for each couple of alternatives (a,b) and for each criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$, the following indices:

- the partial aggregate preference indices:

$$\pi_{\mathbf{r}}(a,b) = \begin{cases} k_{\mathbf{r}} P_{\mathbf{r}}(a,b) & \text{if } \mathbf{r} \in EL, \\ \displaystyle\sum_{\mathbf{t} \in E(G_{\mathbf{r}})} k_{\mathbf{t}} P_{\mathbf{t}}(a,b) & \text{otherwise,} \end{cases}$$

  representing, the degree of preference of a over b, with respect to criterion/subcriterion $G_{\mathbf{r}}$;

- the partial positive, negative, and net outranking flows:

$$\Phi^+_{\mathbf{r}}(a) = \frac{1}{n-1} \sum_{x \in A \setminus \{a\}} \pi_{\mathbf{r}}(a,x), \quad \Phi^-_{\mathbf{r}}(a) = \frac{1}{n-1} \sum_{x \in A \setminus \{a\}} \pi_{\mathbf{r}}(x,a), \quad \Phi_{\mathbf{r}}(a) = \Phi^+_{\mathbf{r}}(a) - \Phi^-_{\mathbf{r}}(a)$$

  representing respectively how strongly alternative a outranks all other alternatives of A on $G_{\mathbf{r}}$, how strongly alternatives of A outrank a on $G_{\mathbf{r}}$, and a balance between the two previous flows.

In this case, we can build preference $P^I_{\mathbf{r}}$, indifference $I^I_{\mathbf{r}}$ and incomparability $R^I_{\mathbf{r}}$ relations of PROMETHEE I as follows:

$$\begin{cases} aP^I_{\mathbf{r}}b & \text{iff} \quad \Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b), \;\; \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b), \text{ and at least one of the two inequalities is strict,} \\ aI^I_{\mathbf{r}}b & \text{iff} \quad \Phi^+_{\mathbf{r}}(a) = \Phi^+_{\mathbf{r}}(b) \text{ and } \Phi^-_{\mathbf{r}}(a) = \Phi^-_{\mathbf{r}}(b), \\ aR^I_{\mathbf{r}}b & \text{otherwise} \end{cases}$$

Moreover, preference $\left(P^{II}_{\mathbf{r}}\right)$ and indifference $\left(I^{II}_{\mathbf{r}}\right)$ relations of PROMETHEE II can be defined as follows:

$$aP^{II}_{\mathbf{r}}b \text{ iff } \Phi_{\mathbf{r}}(a) > \Phi_{\mathbf{r}}(b), \quad \text{while} \quad aI^{II}_{\mathbf{r}}b \text{ iff } \Phi_{\mathbf{r}}(a) = \Phi_{\mathbf{r}}(b).$$

Note 4.1. *Remark that in case* $\mathbf{r} = 0$*, we obtain the indices and relations of the classical PROMETHEE methods for a flat structure of criteria.*

In case of the hierarchy of criteria, we can prove the following Propositions:

Proposition 4.1. *For each* $a, b \in A$*, and for each* $G_{\mathbf{r}} \in \mathcal{G}$*,* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$*, we have:*

1. $\pi_{\mathbf{r}}(a,b) = \sum_{j=1}^{n(\mathbf{r})} \pi_{(\mathbf{r},j)}(a,b)$

2. $\Phi^{+}_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi^{+}_{(\mathbf{r},j)}(a)$

3. $\Phi^{-}_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi^{-}_{(\mathbf{r},j)}(a)$

4. $\Phi_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(a)$

Proof. See Appendix A. □

Proposition 4.2.

1. *Given two alternatives $a, b \in A$ and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that*

$$aP^{I}_{(\mathbf{r},j)}b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aP^{I}_{\mathbf{r}}b$,

2. *Given two alternatives $a, b \in A$ and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that:*

   α) *$\{C_1, C_2\}$ is a partition of the set $\{1, \ldots, n(\mathbf{r})\}$ of indices of subcriteria of $G_{\mathbf{r}}$ in the subsequent level,*

   β) *$aP^{I}_{(\mathbf{r},j)}b$, for all $j \in C_1$,*

   γ) *$aI^{I}_{(\mathbf{r},j)}b$, for all $j \in C_2$,*

   then $aP^{I}_{\mathbf{r}}b$,

3. *Given two alternatives $a, b \in A$ and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that*

$$aI^{I}_{(\mathbf{r},j)}b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aI^{I}_{\mathbf{r}}b$,

Proof. See Appendix A. □

Proposition 4.3.

1. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$*, such that*

$$aP^{II}_{(\mathbf{r},j)}b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aP^{II}_{\mathbf{r}}b$,

2. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$*, such that:*

   α) $\{C_1, C_2\}$ *is a partition of the set* $\{1, \ldots, n(\mathbf{r})\}$ *of indices of subcriteria of* $G_{\mathbf{r}}$ *in the subsequent level,*

   β) $aP^{II}_{(\mathbf{r},j)}b$*, for all* $j \in C_1$,

   γ) $aI^{II}_{(\mathbf{r},j)}b$*, for all* $j \in C_2$,

   then $aP^{II}_{\mathbf{r}}b$,

3. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$*, such that*

$$aI^{II}_{(\mathbf{r},j)}b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

   then $aI^{II}_{\mathbf{r}}b$,

Proof. See Appendix A. □

5 ROR applied to Hierarchical PROMETHEE

5.1 Hierarchical PROMETHEE^{GKS}

In this section we extend the principles of PROMETHEE^{GKS} to the case of the hierarchy of criteria. As stated already above, the only information the DM can obtain from the evaluation matrix is the dominance relation in the set of alternatives. In general, this information is very poor and leaves many alternatives incomparable. To enrich this information, the DM has to introduce some preference information which reveals her/his value system. In this context, we take into account both PROMETHEE I and PROMETHEE II methods, noting that the new Hierarchical PROMETHEE method, so obtained, contains PROMETHEE^{GKS} [13], as a particular case.

Given a subset A^R of A, whose elements are called reference alternatives, and a criterion $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, we suppose that the DM can give two types of preference information regarding $a, b \in A^R$ (we consider $B^R = A^R \times A^R$):

- local relations (denoted by $a \succsim_{\pi_{\mathbf{r}}} b$, $a \succ_{\pi_{\mathbf{r}}} b$, and $a \sim_{\pi_{\mathbf{r}}} b$), comparing directly the performance of a and b on criterion $G_{\mathbf{r}}$, and these comparisons are translated into constraints regarding $\pi_{\mathbf{r}}(a,b)$ and $\pi_{\mathbf{r}}(b,a)$,
- global relations (denoted by $a \succsim_{\Phi_{\mathbf{r}}} b$, $a \succ_{\Phi_{\mathbf{r}}} b$, and $a \sim_{\Phi_{\mathbf{r}}} b$), comparing a and b to all other alternatives, taking into account their outranking flows, $\Phi^{+}_{\mathbf{r}}(a)$, $\Phi^{+}_{\mathbf{r}}(b)$, $\Phi^{-}_{\mathbf{r}}(a)$ and $\Phi^{-}_{\mathbf{r}}(b)$, in case of PROMETHEE I or $\Phi_{\mathbf{r}}(a)$ and $\Phi_{\mathbf{r}}(b)$ in case of PROMETHEE II.

As in the Hierarchical ELECTRE^{GKMS} method, we assume moreover that the DM can give for each elementary subcriterion information regarding indifference and preference thresholds directly, that is provide intervals of possible values, or indirectly, that is provide information on some couples of alternatives (s)he considers indifferent or not (EL_1 and EL_2 represent the sets of criteria for which the DM gives information on the indifference and preference thresholds in a direct or indirect way, respectively); besides, analogously to Hierarchical ELECTRE^{GKMS}, we assume that the DM could provide some information regarding the weights of some elementary subcriterion (for a more detailed description of these preference information and for the consistency constraints on the indifference and preference thresholds see Appendix B).

Given this preference information, a compatible outranking model is a set of preference indices $\pi_{\mathbf{t}}(a,b)$, $(a,b) \in B$, $\mathbf{t} \in EL$, restoring the preference information provided by the DM and satisfying so the following set of constraints (see [13] for a similar formulation in a non-hierarchical case and Appendix B for a detailed description of these constraints):

$$\left.\begin{array}{l}
\text{Pairwise comparisons (local relations), for } (a,b) \in B^R\text{:}\\
\quad \pi_{\mathbf{r}}(a,b) \geq \pi_{\mathbf{r}}(b,a) \text{ if } a \succsim_{\pi_{\mathbf{r}}} b,\\
\quad \pi_{\mathbf{r}}(a,b) \geq \pi_{\mathbf{r}}(b,a) + \varepsilon \text{ if } a \succ_{\pi_{\mathbf{r}}} b,\\
\quad \pi_{\mathbf{r}}(a,b) = \pi_{\mathbf{r}}(b,a) \text{ if } a \sim_{\pi_{\mathbf{r}}} b,\\
\text{Pairwise comparisons (global relations), if the outranking model is exploited in the way}\\
\text{of PROMETHEE II, for } (a,b) \in B^R\text{:}\\
\quad \Phi_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b) \text{ if } a \succsim_{\Phi_{\mathbf{r}}} b,\\
\quad \Phi_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b) + \varepsilon \text{ if } a \succ_{\Phi_{\mathbf{r}}} b,\\
\quad \Phi_{\mathbf{r}}(a) = \Phi_{\mathbf{r}}(b) \text{ if } a \sim_{\Phi_{\mathbf{r}}} b,\\
\text{Pairwise comparisons (global relations), if the outranking model is exploited in the way}\\
\text{of PROMETHEE I:}\\
\quad \Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \;\text{ and }\; \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b) \text{ if } a \succsim_{\Phi_{\mathbf{r}}} b,\; \text{for } (a,b) \in B^R,\\
\quad \left.\begin{array}{l}\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \;\text{ and }\; \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b) \;\text{ and}\\ \Phi^+_{\mathbf{r}}(a) - \Phi^-_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) - \Phi^-_{\mathbf{r}}(b) + \varepsilon\end{array}\right\} \text{ if } \; a \succ_{\Phi_{\mathbf{r}}} b,\; \text{for } (a,b) \in B^R,\\
\quad \Phi^+_{\mathbf{r}}(a) = \Phi^+_{\mathbf{r}}(b) \;\text{ and }\; \Phi^-_{\mathbf{r}}(a) = \Phi^-_{\mathbf{r}}(b) \text{ if } a \sim_{\Phi_{\mathbf{r}}} b,\; \text{for } (a,b) \in B^R.\\
\text{Values of inter-criteria parameters:}\\
\quad \sum_{\mathbf{t} \in EL} \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) = 1,\; \text{where } x_{\mathbf{t},*}, x^*_{\mathbf{t}} \in A \text{ for all } \mathbf{t} \in EL : g_{\mathbf{t}}(x^*_{\mathbf{t}}) = \max_{a \in A} g_{\mathbf{t}}(a),\; \text{and } g_{\mathbf{t}}(x_{\mathbf{t},*}) = \min_{a \in A} g_{\mathbf{t}}(a),\\
\text{Values of marginal preference indices conditioned by intra-criterion preference information,}\\
\text{for all } (a,b) \in B\text{:}\\
\quad k_{\mathbf{t},*} \leq \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) \leq k^*_{\mathbf{t}}, \mathbf{t} \in EL,\\
\quad \pi_{\mathbf{t_1}}(x^*_{\mathbf{t_1}}, x_{\mathbf{t_1},*}) \geq \pi_{\mathbf{t_2}}(x^*_{\mathbf{t_2}}, x_{\mathbf{t_2},*}) + \varepsilon, \;\; \text{if elementary subcriterion } g_{\mathbf{t_1}} \text{ is more important than}\\
\qquad \text{elementary subcriterion } g_{\mathbf{t_2}}, \mathbf{t_1}, \mathbf{t_2} \in EL,\\
\quad \pi_{\mathbf{t_1}}(x^*_{\mathbf{t_1}}, x_{\mathbf{t_1},*}) = \pi_{\mathbf{t_2}}(x^*_{\mathbf{t_2}}, x_{\mathbf{t_2},*}), \;\; \text{if elementary subcriteria } g_{\mathbf{t_1}} \text{ and } g_{\mathbf{t_2}}\\
\qquad \text{are equally important}, \mathbf{t_1}, \mathbf{t_2} \in EL,\\
\quad \pi_{\mathbf{t}}(a,b) = 0 \;\text{ if }\; g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \leq q_{\mathbf{t},*}, \mathbf{t} \in EL_1,\\
\quad \pi_{\mathbf{t}}(a,b) \geq \varepsilon \;\text{ if }\; g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > q^*_{\mathbf{t}}, \mathbf{t} \in EL_1,\\
\quad \pi_{\mathbf{t}}(a,b) + \varepsilon \leq \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) \;\text{ if }\; g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) < p_{\mathbf{t},*}, \mathbf{t} \in EL_1,\\
\quad \pi_{\mathbf{t}}(a,b) = \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) \;\text{ if }\; g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \geq p^*_{\mathbf{t}}, \mathbf{t} \in EL_1,\\
\quad \pi_{\mathbf{t}}(a,b) = 0,\; \pi_{\mathbf{t}}(b,a) = 0 \;\text{ if }\; a \sim_{\mathbf{t}} b,\; \mathbf{t} \in EL_2,\\
\quad \pi_{\mathbf{t}}(a,b) = \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) \;\text{ if }\; a \succ_{\mathbf{t}} b,\; \mathbf{t} \in EL_2.\\
\text{Monotonicity of the functions of marginal preference indices, for all } a,b,c,d \in A, \mathbf{t} \in EL\text{:}\\
\quad \pi_{\mathbf{t}}(a,b) \geq \pi_{\mathbf{t}}(c,d) \;\text{ if }\; g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d),\\
\quad \pi_{\mathbf{t}}(a,b) = \pi_{\mathbf{t}}(c,d) \;\text{ if }\; g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) = g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d).
\end{array}\right\} E^{A^R}$$

If E^{A^R} is feasible and $\varepsilon^* = max\ \varepsilon$, subject to E^{A^R}, is greater than 0, then there exists at least one outranking model compatible with the preference information.

Given a criterion/subcriterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and two alternatives $a, b \in A$, we can give the following definitions:

Definition 5.1.

- *Given a compatible outranking model S exploited in the way of PROMETHEE I, we say a outranks b with respect to $G_{\mathbf{r}}$, and we write $a \succsim_{\mathbf{r}} b$, if:*

$$\Phi^{+}_{\mathbf{r}}(a) \geq \Phi^{+}_{\mathbf{r}}(b) \ \ \textit{and} \ \ \Phi^{-}_{\mathbf{r}}(a) \leq \Phi^{-}_{\mathbf{r}}(b).$$

- *Given a compatible outranking model S exploited in the way of PROMETHEE II, we say that a outranks b with respect to $G_{\mathbf{r}}$, and we write $a \succsim_{\mathbf{r}} b$, if:*

$$\Phi_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b).$$

Note 5.1. *Remark that given two alternatives $a, b \in A$, for each $G_{\mathbf{r}} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and for each compatible outranking model S, if a outranks b with respect to criterion/subcriterion $G_{\mathbf{r}}$ in the sense of PROMETHEE I, then a outranks b with respect to criterion/subcriterion $G_{\mathbf{r}}$ in the sense of PROMETHEE II.*

In the ROR context, considering a criterion/subcriterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and two alternatives $a, b \in A$, we can give the following definitions:

Definition 5.2.

- *a necessarily outranks b with respect to $G_{\mathbf{r}}$, and we write $a \succsim^{N}_{\mathbf{r}} b$, if a outranks b with respect to $G_{\mathbf{r}}$, for all compatible outranking models,*
- *a possibly outranks b with respect to $G_{\mathbf{r}}$, and we write $a \succsim^{P}_{\mathbf{r}} b$, if a outranks b with respect to $G_{\mathbf{r}}$, for at least one compatible outranking model.*

Given a pair of alternatives $(a, b) \in B$, and a criterion/subcriterion $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, necessary and possible outranking relations $(\succsim^{N}_{\mathbf{r}}, \succsim^{P}_{\mathbf{r}})$ can be computed as follows:

- To check whether $a \succsim^{N}_{\mathbf{r}} b$, we assume that a does not outrank b with respect to $G_{\mathbf{r}}$ ($not(a \succsim_{\mathbf{r}} b)$), and we add the corresponding constraints to set E^{A^R} shown below. Then, we verify whether $not(a \succsim_{\mathbf{r}} b)$ is possible in the set of all outranking models compatible with the previously provided preference information.

$$\left.\begin{array}{l}
E^{A^R} \\[2ex]
\text{if one verifies the truth of global outranking:} \\[2ex]
\quad \text{if exploited in the way of PROMETHEE II, then:} \\[2ex]
\qquad \Phi_{\mathbf{r}}(a) + \varepsilon \leq \Phi_{\mathbf{r}}(b) \\[2ex]
\quad \text{if exploited in the way of PROMETHEE I, then:} \\[2ex]
\qquad \Phi^{+}_{\mathbf{r}}(a) + \varepsilon \leq \Phi^{+}_{\mathbf{r}}(b) + 2M^{0}_{\mathbf{r}} \;\; \text{and} \;\; \Phi^{-}_{\mathbf{r}}(a) + 2M^{1}_{\mathbf{r}} \geq \Phi^{-}_{\mathbf{r}}(b) + \varepsilon \\[2ex]
\qquad\quad \text{where } M^{\mathbf{r}}_{i} \in \{0,1\}, \;\; i = 1,2, \;\; \text{and} \;\; \sum_{i=0}^{1} M^{\mathbf{r}}_{i} \leq 1 \\[2ex]
\text{if one verifies the truth of local outranking:} \\[2ex]
\qquad \pi_{\mathbf{r}}(a,b) + \varepsilon \leq \pi_{\mathbf{r}}(b,a)
\end{array}\right\} E^{N}_{\mathbf{r}}(a,b)$$

We say that:

$$a \succsim^{N} b \;\text{ if }\; E^{N}_{\mathbf{r}}(a,b) \;\text{ is infeasible or }\; \varepsilon^{N}_{\mathbf{r}}(a,b) \leq 0, \;\;\text{ where }\; \varepsilon^{N}_{\mathbf{r}}(a,b) = max\ \varepsilon, \;\;\text{ subject to }\; E^{N}_{\mathbf{r}}(a,b).$$

Observe that in $E^{N}_{\mathbf{r}}(a,b)$, the binary variables $M^{0}_{\mathbf{r}}$ and $M^{1}_{\mathbf{r}}$ are used in order to deny the outranking of a over b. In fact, a does not outrank b if $\Phi^{+}_{\mathbf{r}}(a) < \Phi^{+}_{\mathbf{r}}(b)$ or $\Phi^{-}_{\mathbf{r}}(a) > \Phi^{-}_{\mathbf{r}}(b)$. If $M^{i}_{\mathbf{r}} = 0, i = 0,1$, then the corresponding constraint opposes a veto to the outranking of a over b (in particular, if $M^{0}_{\mathbf{r}} = 0$ then $\Phi^{+}_{\mathbf{r}}(a) < \Phi^{+}_{\mathbf{r}}(b)$ while if $M^{1}_{\mathbf{r}} = 0$ then $\Phi^{-}_{\mathbf{r}}(a) > \Phi^{-}_{\mathbf{r}}(b)$); instead, if $M^{i}_{\mathbf{r}} = 1, i = 0,1$, the corresponding constraint is always verified reminding that $\Phi^{+}_{\mathbf{r}}(a) \in [0,1]$ and $\Phi^{-}_{\mathbf{r}}(a) \in [0,1]$ for all $a \in A$. Besides, the constraint $\sum_{i=0}^{1} M^{i}_{\mathbf{r}} \leq 1$ ensures that at least one of the two variables has to be equal to zero.

- To check whether $a \succsim^{P}_{\mathbf{r}} b$, we assume that a outranks b with respect to $G_{\mathbf{r}}$ ($a \succsim_{\mathbf{r}} b$), and add corresponding constraints to the set E^{A^R} shown below. Then, we verify whether $a \succsim_{\mathbf{r}} b$ is possible in the set of all compatible outranking models.

$$\left.\begin{array}{l} E^{A^R} \\ \text{if one verifies the truth of global outranking:} \\ \quad \text{if exploited in the way of PROMETHEE II, then:} \\ \quad\quad \Phi_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b) \\ \quad \text{if exploited in the way of PROMETHEE I, then:} \\ \quad\quad \Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \ \text{ and } \ \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b) \\ \text{if one verifies the truth of local outranking:} \\ \quad\quad \pi_{\mathbf{r}}(a,b) \geq \pi_{\mathbf{r}}(b,a) \end{array}\right\} E^P_{\mathbf{r}}(a,b)$$

We say that:

$a \succsim^P_{\mathbf{r}} b$ if $E^P_{\mathbf{r}}(a,b)$ is feasible and $\varepsilon^P_{\mathbf{r}}(a,b) > 0$, where $\varepsilon^P_{\mathbf{r}}(a,b) = max\ \varepsilon$, subject to $E^P_{\mathbf{r}}(a,b)$.

Note 5.2. *The same observation made for ELECTRE about application of linear programming within ROR is valid for PROMETHEE. More precisely, if the DM is able to give the marginal function* $P_{\mathbf{t}}(a,b)$ *and the related thresholds, the ROR optimization problems can be formulated in terms of linear programming, taking into account as variables the weights* $k_{\mathbf{t}}, \mathbf{t} \in EL$, *only. This amounts to substitute the set of constraints* E^{A^R} *with the following:*

$$\left.\begin{array}{l}
\textit{Pairwise comparisons (local relations), for } (a,b) \in B^R\textit{:} \\
\quad \pi_{\mathbf{r}}(a,b) \geq \pi_{\mathbf{r}}(b,a) \textit{ if } a \succsim_{\pi_{\mathbf{r}}} b, \\
\quad \pi_{\mathbf{r}}(a,b) \geq \pi_{\mathbf{r}}(b,a) + \varepsilon \textit{ if } a \succ_{\pi_{\mathbf{r}}} b, \\
\quad \pi_{\mathbf{r}}(a,b) = \pi_{\mathbf{r}}(b,a) \textit{ if } a \sim_{\pi_{\mathbf{r}}} b, \\
\textit{Pairwise comparisons (global relations), if the outranking model is exploited in the way} \\
\textit{of PROMETHEE II, for } (a,b) \in B^R\textit{:} \\
\quad \Phi_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b) \textit{ if } a \succsim_{\Phi_{\mathbf{r}}} b, \\
\quad \Phi_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b) + \varepsilon \textit{ if } a \succ_{\Phi_{\mathbf{r}}} b, \\
\quad \Phi_{\mathbf{r}}(a) = \Phi_{\mathbf{r}}(b) \textit{ if } a \sim_{\Phi_{\mathbf{r}}} b, \\
\textit{Pairwise comparisons (global relations), if the outranking model is exploited in the way} \\
\textit{of PROMETHEE I:} \\
\quad \Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \;\; \textit{and} \;\; \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b) \textit{ if } a \succsim_{\Phi_{\mathbf{r}}} b, \textit{ for } (a,b) \in B^R, \\
\quad \left.\begin{array}{l} \Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \;\; \textit{and} \;\; \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b) \;\; \textit{and} \\ \Phi^+_{\mathbf{r}}(a) - \Phi^-_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) - \Phi^-_{\mathbf{r}}(b) + \varepsilon \end{array}\right\} \textit{ if } a \succ_{\Phi_{\mathbf{r}}} b, \textit{ for } (a,b) \in B^R, \\
\quad \Phi^+_{\mathbf{r}}(a) = \Phi^+_{\mathbf{r}}(b) \;\; \textit{and} \;\; \Phi^-_{\mathbf{r}}(a) = \Phi^-_{\mathbf{r}}(b) \textit{ if } a \sim_{\Phi_{\mathbf{r}}} b, \textit{ for } (a,b) \in B^R. \\
\textit{Values of marginal preference indices conditioned by intra-criterion preference information,} \\
\textit{for all } (a,b) \in B\textit{:} \\
\quad k_{\mathbf{t},*} \leq k_{\mathbf{t}} \leq k^*_{\mathbf{t}}, \mathbf{t} \in EL, \\
\quad k_{\mathbf{t_1}} \geq k_{\mathbf{t_2}} + \varepsilon, \;\; \textit{if elementary subcriterion } g_{\mathbf{t_1}} \textit{ is more important than} \\
\qquad \textit{elementary subcriterion } g_{\mathbf{t_2}}, \mathbf{t_1}, \mathbf{t_2} \in EL, \\
\quad k_{\mathbf{t_1}} = k_{\mathbf{t_2}}, \;\; \textit{if elementary subcriteria } g_{\mathbf{t_1}} \textit{ and } g_{\mathbf{t_2}} \\
\qquad \textit{are equally important}, \mathbf{t_1}, \mathbf{t_2} \in EL,
\end{array}\right\} E^{A^R}$$

5.2 Properties of necessary and possible outranking relations

Proposition 5.1.

1. *For all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $\succsim^N_{\mathbf{r}} \subseteq \succsim^P_{\mathbf{r}}$,

2. *For all* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $\succsim^P_{\mathbf{r}}$ *and* $\succsim^N_{\mathbf{r}}$ *are reflexive,*

Proof. See Appendix A. □

Proposition 5.2.

1. *Given two alternatives* $a, b \in A$ *and* $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (EL \cup LBO)$, *such that:*

$$a \succsim^N_{(\mathbf{r},j)} b \quad \textit{for all } j = 1, \ldots, n(\mathbf{r}),$$

then $a \succsim^N_{\mathbf{r}} b$.

2. Given two alternatives $a, b \in A$ and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus (EL \cup LBO)$, such that:

α) $a \succsim^N_{(\mathbf{r},j)} b$ *for all* $j = 1, \ldots, n(\mathbf{r}), j \neq w$,

β) $a \succsim^P_{(\mathbf{r},w)} b$,

then $a \succsim^P_{\mathbf{r}} b$.

Proof. See Appendix A. □

5.3 An illustrative example

In this subsection we consider the same problem we have dealt with Hierarchical ELECTRE in subsection 3.3 using both PROMETHEE and PROMETHEE^{GKS} methods extended to the case of a hierarchical family of criteria.

At first, we suppose to have the same weights, as well as the same indifference and preference thresholds as before: let us also choose for each elementary subcriterion $g_{\mathbf{t}}$, $\mathbf{t} \in EL$, the following preference function $P_{\mathbf{t}}(a, b)$, for any $a, b \in A$:

$$P_{\mathbf{t}}(a,b) = \begin{cases} 0 & \text{if} \quad g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \leq q_{\mathbf{t}}, \\ \frac{g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) - q_{\mathbf{t}}}{p_{\mathbf{t}} - q_{\mathbf{t}}} & \text{if} \quad q_{\mathbf{t}} < g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) < p_{\mathbf{t}}, \\ 1 & \text{if} \quad g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \geq p_{\mathbf{t}}. \end{cases}$$

Table 6: Preference relations obtained using Hierarchical PROMETHEE I

Comprehensively	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P	$\underline{P}$
s_2	P^{-1}	I	P	P^{-1}	P
s_3	P^{-1}	P^{-1}	I	P^{-1}	P
s_4	P^{-1}	P	P	I	P
s_5	P^{-1}	P	R	R	I

Maths	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P	R
s_2	P^{-1}	I	P^{-1}	P^{-1}	R
s_3	P^{-1}	P	I	R	R
s_4	P^{-1}	P	R	I	R
s_5	R	R	R	R	I

Chemistry	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P^{-1}	P
s_2	P^{-1}	I	P	P^{-1}	P
s_3	P^{-1}	P^{-1}	I	P^{-1}	P
s_4	P	P	P	I	P
s_5	P^{-1}	P^{-1}	P^{-1}	P^{-1}	I

Algebra	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P	R
s_2	P^{-1}	I	P^{-1}	P^{-1}	R
s_3	P^{-1}	P	I	R	R
s_4	P^{-1}	P	R	I	R
s_5	P^{-1}	P^{-1}	P^{-1}	P^{-1}	I

Analysis	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P	P
s_2	P^{-1}	I	P^{-1}	P^{-1}	P^{-1}
s_3	P^{-1}	P	I	R	R
s_4	P^{-1}	P	R	I	R
s_5	P^{-1}	P	R	R	I

Anal. Chem.	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P^{-1}	P
s_2	P^{-1}	I	P	P^{-1}	P
s_3	P^{-1}	P^{-1}	I	P^{-1}	P
s_4	P	P	P	I	P
s_5	P^{-1}	P^{-1}	P^{-1}	P^{-1}	I

Org. Chem.	s_1	s_2	s_3	s_4	s_5
s_1	I	P	P	P^{-1}	P
s_2	P^{-1}	I	P	P^{-1}	P
s_3	P^{-1}	P^{-1}	I	P^{-1}	P
s_4	P	P	P	I	P
s_5	P^{-1}	P^{-1}	P^{-1}	P^{-1}	I

In Table 6 we present the preference relations that PROMETHEE I states for any level of the considered hierarchy of criteria. More precisely, considering $Matrix_{\mathbf{r}}$ to be one of the seven matrices presented in Table 6, we have:

$$Matrix_{\mathbf{r}}(s_i,s_j)=\begin{cases} P & \text{if} \quad s_i \text{ is preferred to } s_j \text{ with respect to criterion } G_{\mathbf{r}}, \\ I & \text{if} \quad s_i \text{ is indifferent to } s_j \text{ with respect to criterion } G_{\mathbf{r}}, \\ R & \text{if} \quad s_i \text{ is incomparable to } s_j \text{ with respect to criterion } G_{\mathbf{r}}, \\ P^{-1} & \text{if} \quad s_j \text{ is preferred to } s_i \text{ with respect to criterion } G_{\mathbf{r}} \end{cases}$$

In Table 6, $Matrix_{Comprehensively}(s_1,s_5)=P$ is underlined in order to evidence that there exists a couple of alternatives (s_i,s_j) such that s_i is preferred to s_j with respect to some criterion $G_{\mathbf{r}}$, but with respect to a subcriterion immediately descending from $G_{\mathbf{r}}$, say $G_{(\mathbf{r},\mathbf{w})}$, s_i is not preferred to s_j. In our example, s_1 is preferred to s_5 with respect to the totality of criteria, but s_1 is incomparable to s_5 with respect to Mathematics being a subcriterion of the totality of criteria. Note that the underlined couple (s_1,s_5) is not the only example of such a situation in Table 6.

In Table 7, we can see the ranking obtained by Hierarchical PROMETHEE II for each criterion/subcriterion of the hierarchy.

Table 7: Ranking of students at all levels of the hierarchy of criteria, obtained using Hierarchical PROMETHEE II

Position/subject	**Comprehensive**	**Maths**	**Algebra**	**Analysis**	**Chemistry**	**Analytical Chemistry**	**Organic Chemistry**
1	s_1 (0.2000)	s_1 (0.0938)	s_1 (0.0417)	s_1 (0.0521)	s_4 (0.1250)	s_4 (0.0500)	s_4 (0.0750)
2	s_4 (0.1062)	s_3 (0.0375)	s_3 (0.0125)	s_3 (0.0250)	s_1 (0.1063)	s_1 (0.0396)	s_1 (0.0667)
3	s_2 (-0.0167)	s_4 (-0.0187)	s_5 (-0.0083)	s_4 (-0.0021)	s_2 (0.0688)	s_2 (0.0188)	s_2 (0.0500)
4	s_3 (-0.0938)	s_5 (-0.0271)	s_4 (-0.0167)	s_5 (-0.0188)	s_3 (-0.1313)	s_3 (-0.0438)	s_3 (-0.0875)
5	s_5 (-0.1958)	s_2 (-0.0854)	s_2 (-0.0292)	s_2 (-0.0563)	s5 (-0.1688)	s_5 (-0.0646)	s_5 (-0.1042)

Now, let us suppose that the Dean decides to use the Hierarchical PROMETHEE^{GKS} providing some detailed outranking and non-outranking information with respect to all criteria considered together, and with respect to particular subcriteria. At the same time, (s)he wishes to obtain detailed information regarding the necessary and possible outranking relations. In order to use the methodology presented above, we suppose that the Dean can give information regarding indifference and preference thresholds on all elementary subcriteria, as shown in Table 8.

Table 8: Indifference and preference thresholds provided by the DM

Elementary subcriterion, $g_{\mathbf{t}}$	$q_{\mathbf{t},*}$	$q_{\mathbf{t}}^*$	$p_{\mathbf{t},*}$	$p_{\mathbf{t}}^*$
Group Theory	1	2	4	5
Linear Algebra	1	2	4	5
Calculus	1	2	4	5
Functions Theory	1	2	4	5
Analytical Chemistry I	1	2	4	5
Applied Analytical Chemistry	1	2	4	5
Organic Chemistry I	1	2	4	5
Organic Chemistry II	1	2	4	5

Let us first suppose, that the outranking relation is exploited in the way of PROMETHEE II, and that the Dean gives the following preference information:

- with respect to Mathematics, student s_4 is preferred to each other student more than student s_2 is preferred to each other student ($s_4 \succ_{\Phi_{\mathbf{1}}} s_2$),
- with respect to Organic Chemistry, student s_4 is preferred to each other student more than student s_3 is preferred to each other student ($s_4 \succ_{\Phi_{(\mathbf{2},\mathbf{2})}} s_3$).

These two pieces of information are translated into the following constraints regarding the variables of the ordinal regression problem:

- $s_4 \succ_{\Phi_{\mathbf{1}}} s_2$ is translated into:

$$\Phi_{\mathbf{1}}(s_4) = \sum_{x \in A \setminus \{s_4\}} \left\{ \sum_{\mathbf{t} \in E(G_{\mathbf{1}})} \frac{1}{n-1} \pi_{\mathbf{t}}(s_4, x) \right\} - \sum_{x \in A \setminus \{s_4\}} \left\{ \sum_{\mathbf{t} \in E(G_{\mathbf{1}})} \frac{1}{n-1} \pi_{\mathbf{t}}(x, s_4) \right\} \geq$$

$$\geq \Phi_{\mathbf{1}}(s_2) = \sum_{x \in A \setminus \{s_2\}} \left\{ \sum_{\mathbf{t} \in E(G_{\mathbf{1}})} \frac{1}{n-1} \pi_{\mathbf{t}}(s_2, x) \right\} - \sum_{x \in A \setminus \{s_2\}} \left\{ \sum_{\mathbf{t} \in E(G_{\mathbf{1}})} \frac{1}{n-1} \pi_{\mathbf{t}}(x, s_2) \right\} + \varepsilon$$

- $s_4 \succ_{\Phi_{(\mathbf{2},\mathbf{2})}} s_3$ is translated into:

$$\Phi_{(\mathbf{2},\mathbf{2})}(s_4) = \sum_{x \in A \setminus \{s_4\}} \left\{ \sum_{\mathbf{t} \in E(G_{(\mathbf{2},\mathbf{2})})} \frac{1}{n-1} \pi_{\mathbf{t}}(s_4, x) \right\} - \sum_{x \in A \setminus \{s_4\}} \left\{ \sum_{\mathbf{t} \in E(G_{(\mathbf{2},\mathbf{2})})} \frac{1}{n-1} \pi_{\mathbf{t}}(x, s_4) \right\} \geq$$

$$\geq \Phi_{(\mathbf{2},\mathbf{2})}(s_3) = \sum_{x \in A \setminus \{s_3\}} \left\{ \sum_{\mathbf{t} \in E(G_{(\mathbf{2},\mathbf{2})})} \frac{1}{n-1} \pi_{\mathbf{t}}(s_3, x) \right\} - \sum_{x \in A \setminus \{s_3\}} \left\{ \sum_{\mathbf{t} \in E(G_{(\mathbf{2},\mathbf{2})})} \frac{1}{n-1} \pi_{\mathbf{t}}(x, s_3) \right\} + \varepsilon$$

In Table 9, we show the necessary outranking relation with respect to some criteria and subcriteria of the hierarchy. For other subcriteria the necessary outranking relation is empty.

Table 9: Necessary outranking relations obtained using Hierarchical PROMETHEE^{GKS} and exploitation of the outranking relation in the way of PROMETHEE II

$\succsim^N_{(\mathbf{0})}$	s_1	s_2	s_3	s_4	s_5	$\succsim^N_{(\mathbf{1})}$	s_1	s_2	s_3	s_4	s_5	$\succsim^N_{(\mathbf{1},\mathbf{2})}$	s_1	s_2	s_3	s_4	s_5	$\succsim^N_{(\mathbf{2},\mathbf{2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	0	0	0	0	s_1	1	1	0	0	0	s_1	1	0	0	0	0	s_1	1	0	0	0	0
s_2	0	1	0	0	0	s_2	0	1	0	0	0	s_2	0	1	0	0	0	s_2	0	1	0	0	0
s_3	0	0	1	0	0	s_3	0	1	1	0	0	s_3	0	0	1	0	0	s_3	0	0	1	0	1
s_4	0	0	0	1	0	s_4	0	1	0	1	0	s_4	0	0	0	1	0	s_4	0	0	1	1	1
s_5	0	0	0	0	1	s_5	0	0	0	0	1	s_5	0	1	0	0	1	s_5	0	0	0	0	1

In the first matrix of Table 9, we observe that preference information provided by the DM does not imply any necessary outranking with respect to the totality of criteria. At the same time, we obtain partial information that cannot be obtained by PROMETHEE^{GKS} for a flat structure of the set of criteria; for example, we can see that students s_1, s_3 and s_4 are necessarily preferred to student s_2 with respect to Mathematics, so as student s_4 is necessarily preferred to student s_3 and s_5 with respect to Organic Chemistry, and so on.

Now, let us suppose that the outranking relation is exploited in the way of PROMETHHE I. We are considering the same preference information provided by the Dean. It is translated, however, in a different way than before:

- $s_4 \succ_{\Phi_{\mathbf{1}}} s_2$ is translated into constraints:

1. $\Phi^+_{(\mathbf{1})}(s_4) \geq \Phi^+_{(\mathbf{1})}(s_2) \Leftrightarrow \displaystyle\sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_4,x)\right\} \geq \sum_{x\in A\setminus\{s_2\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_2,x)\right\}$,
2. $\Phi^-_{(\mathbf{1})}(s_4) \leq \Phi^-_{\mathbf{1}}(s_2) \Leftrightarrow \displaystyle\sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_4)\right\} \leq \sum_{x\in A\setminus\{s_2\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_2)\right\}$,
3. $\Phi^+_{(\mathbf{1})}(s_4) - \Phi^-_{(\mathbf{1})}(s_4) \geq \Phi^+_{(\mathbf{1})}(s_2) - \Phi^-_{(\mathbf{1})}(s_2) + \varepsilon \Leftrightarrow$
$$\sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_4,x)\right\} - \sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_4)\right\} \geq$$
$$\geq \sum_{x\in A\setminus\{s_2\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_2,x)\right\} - \sum_{x\in A\setminus\{s_2\}}\left\{\sum_{\mathbf{t}\in E(G_{\mathbf{1}})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_2)\right\} + \varepsilon.$$

- $s_4 \succ_{\Phi_{(\mathbf{2},\mathbf{2})}} s_3$ is translated into constraints:

1. $\Phi^+_{(\mathbf{2},\mathbf{2})}(s_4) \geq \Phi^+_{(\mathbf{2},\mathbf{2})}(s_3) \Leftrightarrow \displaystyle\sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_4,x)\right\} \geq \sum_{x\in A\setminus\{s_3\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_3,x)\right\}$,
2. $\Phi^-_{(\mathbf{2},\mathbf{2})}(s_4) \leq \Phi^-_{(\mathbf{2},\mathbf{2})}(s_3) \Leftrightarrow \displaystyle\sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_4)\right\} \leq \sum_{x\in A\setminus\{s_3\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_3)\right\}$,
3. $\Phi^+_{(\mathbf{2},\mathbf{2})}(s_4) - \Phi^-_{(\mathbf{2},\mathbf{2})}(s_4) \geq \Phi^+_{(\mathbf{2},\mathbf{2})}(s_3) - \Phi^-_{(\mathbf{2},\mathbf{2})}(s_3) + \varepsilon \Leftrightarrow$
$$\sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_4,x)\right\} - \sum_{x\in A\setminus\{s_4\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_4)\right\} \geq$$
$$\geq \sum_{x\in A\setminus\{s_3\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(s_3,x)\right\} - \sum_{x\in A\setminus\{s_3\}}\left\{\sum_{\mathbf{t}\in E(G_{(\mathbf{2},\mathbf{2})})}\frac{1}{n-1}\pi_{\mathbf{t}}(x,s_3)\right\} + \varepsilon.$$

In Table 10, we show the necessary outranking relation with respect to some subcriteria of the hierarchy. Also in this case, the necessary outranking relation with respect to the totality of criteria is empty, however, it is interesting to see some partial information at lower levels of the hierarchy, where the necessary outranking relation is not empty; e.g: with respect to Analysis student s_5 necessarily outranks student s_2, or with respect to Organic Chemistry, student s_3 necessarily outranks student s_5, and so on.

Table 10: Necessary outranking relations obtained using Hierarchical PROMETHEE^{GKS} and exploitation of the outranking relation in the way of PROMETHEE I

$\succsim^N_{(\mathbf{0})}$	s_1	s_2	s_3	s_4	s_5	$\succsim^N_{(\mathbf{1,2})}$	s_1	s_2	s_3	s_4	s_5	$\succsim^N_{(\mathbf{2,2})}$	s_1	s_2	s_3	s_4	s_5
s_1	1	0	0	0	0	s_1	1	0	0	0	0	s_1	1	0	0	0	0
s_2	0	1	0	0	0	s_2	0	1	0	0	0	s_2	0	1	0	0	0
s_3	0	0	1	0	0	s_3	0	0	1	0	0	s_3	0	0	1	0	1
s_4	0	0	0	1	0	s_4	0	0	0	1	0	s_4	0	0	0	1	0
s_5	0	0	0	0	1	s_5	0	1	0	0	1	s_5	0	0	0	0	1

6 Conclusions

In this paper, we proposed a new procedure aiming at extending the outranking methods to the case of the hierarchy of criteria in the way introduced in [3]. The family of criteria is not considered at the same level, but, instead, it has a hierarchical structure. Considering the hierarchical structure of criteria, the Decision Maker (DM) can obtain not only comprehensive preference relation with respect to all criteria, but also partial preference relation with respect to subcriteria at different levels of the hierarchy. This is not possible when considering the flat structure of criteria.

Let us remark that the use of the hierarchy of criteria proposed by our approach is rather different from other MCDA methodologies [19, 5]. In fact, while in general the hierarchy of criteria is used to decompose and make easier the preference elicitation concerning pairwise comparisons of criteria with respect to relative importance, in our approach, a preference relation in each node of the hierarchy constitutes a base for discussion with the DM.

We wish to stress that this specific use of the hierarchy of criteria can be applied to any MCDA methodology. In this paper we have applied it to Robust Ordinal Regression (ROR) approach, but it can be applied to any other MCDA methodology, even those which use the hierarchy to ask the DM for pairwise comparisons of subcriteria with respect to their importance.

Remark, moreover, that our hierarchical procedures boil down to the classical ELECTRE and PROMETHEE methods when criteria are considered at one level only. This proves that our hierarchical procedures generalize the classical outranking methods.

We presented the hierarchical outranking methods for two types of preference information from the part of the DM: direct, considered in classical outranking methods, and indirect, considered in Robust Ordinal Regression for outranking methods. ROR takes into account all outranking models compatible with preference information provided by the DM in terms of exemplary outranking and non-outranking relations for some pairs of reference alternatives. It is producing two binary relations: the necessary outranking relation (S^N, $\succsim^N$), for which a outranks b for all compatible outranking models, and the possible outranking relation (S^P, $\succsim^P$), for which a outranks b for at least one compatible outranking model. When ROR is applied to hierarchical outranking methods, one gets necessary ($\succsim_{\mathbf{r}}^N$) and possible ($\succsim_{\mathbf{r}}^P$) outranking relations for each criterion/subcriterion $G_{\mathbf{r}}$ belonging to the hierarchy. In this way, the DM knows the necessary and possible preference relations for given preference information, not only at the comprehensive level, for the totality of criteria, but also for any criterion/subcriterion of the hierarchy. Such finer information about preferences has an advantage over the comprehensive information because it permits to decompose the comprehensive preferences into their constituent elements. The application of ROR to ELECTRE and PROMETHEE methods was done in [9] and [13], but also in this case, our hierarchical procedures can be considered as generalizations of both ELECTRE^{GKMS} and PROMETHEE^{GKS} because the hierarchical procedures boil down to these methods when all criteria are considered at the same level. In a companion paper, we propose to extend the hierarchy of criteria also on the sorting outranking methods.

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References

[1] S. Angilella, S. Greco, and B. Matarazzo. Non-additive robust ordinal regression: A multiple criteria decision model based on the choquet integral. *European Journal of Operational Research*, 201(1):277–288, 2010.

[2] J.P Brans, B. Mareschal, and Ph. Vincke. PROMETHEE: a new family of outranking methods in multicriteria analysis. In J.P Brans, editor, *Operational Research, IFORS 84*, pages 477–490. North Holland, Amsterdam, 1984.

[3] S. Corrente, S. Greco, and R. Słowiński. Multiple Criteria Hierarchy Process in Robust Ordinal Regression. *Decision Support Systems*, 53(3):660–674, 2012.

[4] M. Ehrgott, J. Figueira, and S. Greco (eds.). *Trends in Multiple Criteria Decision Analysis*. Springer, Berlin, 2010.

[5] D. Ergu, G. Kou, Y. Peng, and Y. Shi. A Simple Method to Improve the Consistency Ratio of the Pair-wise Comparison Matrix in ANP. *European Journal of Operational Research*, 213(1):246–259, 2011.

[6] J. Figueira, S. Greco, and M. Ehrgott (eds.). *Multiple Criteria Decision Analysis: State of the Art Surveys*. Springer, Berlin, 2005.

[7] J.R. Figueira, S. Greco, B. Roy, and R. Słowiński. An overview of ELECTRE methods and their recent extensions. *Journal of Multicriteria Decision Analysis, To appear*, 2012, DOI: 10.1002/mcda.1482.

[8] J.R. Figueira, S. Greco, and R. Słowiński. Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. *European Journal of Operational Research*, 195(2):460–486, 2009.

[9] S. Greco, M. Kadziński, V.Mousseau, and R. Słowiński. ELECTRE^{GKMS}: Robust ordinal regression for outranking methods. *European Journal of Operational Research*, 214(1):118–135, 2011.

[10] S. Greco, V. Mousseau, and R. Słowiński. Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. *European Journal of Operational Research*, 191(2):416–436, 2008.

[11] S. Greco, V. Mousseau, and R. Słowiński. Multiple criteria sorting with a set of additive value functions. *European Journal of Operational Research*, 207(3):1455–1470, 2010.

[12] E. Jacquet-Lagrèze and J. Siskos. Assessing a set of additive utility functions for multicriteria decision-making, the UTA method. *European journal of operational research*, 10(2):151–164, 1982.

[13] M. Kadziński, S. Greco, and R. Słowiński. Extreme ranking analysis in robust ordinal regression. *Omega*, 40(4):488–501, 2012.

[14] R.L. Keeney and H. Raiffa. *Decisions with multiple objectives: Preferences and value tradeoffs*. J. Wiley, New York, 1976.

[15] D. Pekelman and S.K. Sen. Mathematical programming models for the determination of attribute weights. *Management Science*, 20(8):1217–1229, 1974.

[16] B. Roy. *Multicriteria Methodology for Decision Aiding*. Kluwer, Dordrecht, 1996.

[17] B. Roy. ELECTRE Methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 133–162. Springer, Berlin, 2005.

[18] B. Roy and D. Bouyssou. *Aide multicritère à la décision: méthodes et cas*. Economica, 1993.

[19] T. L. Saaty. The analytic hierarchy and analytic network processes for the measurement of intangible criteria and for decision-making. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 345–382. Springer, Berlin, 2005.

[20] V. Srinivasan and A.D. Shocker. Estimating the weights for multiple attributes in a composite criterion using pairwise judgments. *Psychometrika*, 38(4):473–493, 1973.

Appendix A

Proof of Proposition 2.1

1. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, and $a, b \in A$, such that $aS_{(\mathbf{r},j)}b$, for all $j = 1, \ldots, n(\mathbf{r})$. This means that:

   $\alpha)$ $C_{(\mathbf{r},j)}(a,b) \geq \lambda_{(\mathbf{r},j)}$, for all $j = 1, \ldots, n(\mathbf{r})$,

   $\beta)$ $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) < v_{\mathbf{t}}$, for all $\mathbf{t} \in E(G_{(\mathbf{r},j)})$, for all $j = 1, \ldots, n(\mathbf{r})$.

   Noting that we are considering the case in which each criterion belongs to only one of the criteria from the upper level (see Section 2.1), we have:

   $\gamma)$ $\cup_{j=1}^{n(\mathbf{r})} E(G_{(\mathbf{r},j)}) = E(G_{\mathbf{r}})$,

   $\delta)$ $C_{\mathbf{r}}(a,b) = \displaystyle\sum_{j=1}^{n(\mathbf{r})} C_{(\mathbf{r},j)}(a,b)$,

   $\theta)$ $\lambda_{\mathbf{r}} = \displaystyle\sum_{j=1}^{n(\mathbf{r})} \lambda_{(\mathbf{r},j)}$,

   thus
   $$C_{\mathbf{r}}(a,b) = \sum_{j=1}^{n(\mathbf{r})} C_{(\mathbf{r},j)}(a,b) \geq \sum_{j=1}^{n(\mathbf{r})} \lambda_{(\mathbf{r},j)} = \lambda_{\mathbf{r}} \qquad \text{by } \delta), \ \alpha) \text{ and } \theta)$$

   and
   $$g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) < v_{\mathbf{t}}, \text{ for all } \mathbf{t} \in E(G_{\mathbf{r}}) \qquad \text{by } \beta) \text{ and } \gamma).$$

   This implies that $aS_{\mathbf{r}}b$.

2. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, and $a, b \in A$, such that $not(aS_{(\mathbf{r},j)}b)$, for all $j = 1, \ldots, n(\mathbf{r})$. This means that for all $j = 1, \ldots, n(\mathbf{r})$ we have:

   $\alpha')$ $C_{(\mathbf{r},j)}(a,b) < \lambda_{(\mathbf{r},j)}$ or

   $\beta')$ $\exists \mathbf{t} \in E(G_{(\mathbf{r},j)}) : g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq v_{\mathbf{t}}$.

We distinguish two cases:

- Let us suppose that $not(aS_{(\mathbf{r},j)}b)$ is satisfied because of $\beta^{'}$); thus there exists one elementary subcriterion $g_{\mathbf{t}} \in E(G_{(\mathbf{r},j)})$ such that $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq v_{\mathbf{t}}$; being $E(G_{(\mathbf{r},j)}) \subseteq E(G_{\mathbf{r}})$, $g_{\mathbf{t}}$ is an elementary subcriterion belonging also to $E(G_{\mathbf{r}})$ and so it opposes veto to the outranking of a over b with respect to criterion $G_{\mathbf{r}}$; therefore $not(aS_{\mathbf{r}}b)$.

- Let us suppose that for all $j = 1, \ldots, n(\mathbf{r})$, $\beta^{'}$) is never satisfied, that is for all $j = 1, \ldots, n(\mathbf{r})$, for all $\mathbf{t} \in E(G_{(\mathbf{r},j)})$, $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) < v_{\mathbf{t}}$. Thus, for all $j = 1, \ldots, n$, $not(aS_{(\mathbf{r},j)}b)$ holds because of $\alpha^{'}$), that is for all $j = 1, \ldots, n(\mathbf{r})$, $C_{(\mathbf{r},j)}(a,b) < \lambda_{(\mathbf{r},j)}$. Reminding $\gamma), \delta)$ and $\theta)$ of point 1. of this Proposition and by $\alpha^{'}$) we have:

$$C_{\mathbf{r}}(a,b) = \sum_{j=1}^{n(\mathbf{r})} C_{(\mathbf{r},j)}(a,b) < \sum_{j=1}^{n(\mathbf{r})} \lambda_{(\mathbf{r},j)} = \lambda_{\mathbf{r}}.$$

  This opposes to outranking of a over b with respect to criterion $G_{\mathbf{r}}$, and thus $not(aS_{\mathbf{r}}b)$.

Proof of Proposition 3.1

1. Let $a, b \in A$, and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ such that $aS^{N}_{\mathbf{r}}b$. This means that $aS_{\mathbf{r}}b$ for all compatible outranking models, and thus there exists at least one compatible outranking model for which $aS_{\mathbf{r}}b$, thus $aS^{P}_{\mathbf{r}}b$.

2. Let S an outranking relation, $a \in A$ an alternative and $G_{\mathbf{r}}$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ a criterion/subcriterion. We have that:

   - for all $\mathbf{t} \in E(G_{\mathbf{r}})$, $\phi_{\mathbf{t}}(a,a) = 1$, and therefore by equation (1), $C_{\mathbf{r}}(a,a) = K_{\mathbf{r}}$,
   - for all $\lambda_{\mathbf{r}} \in \left[\frac{K_{\mathbf{r}}}{2}, K_{\mathbf{r}}\right]$, $C_{\mathbf{r}}(a,a) \geq \lambda_{\mathbf{r}}$, (it follows by previous point),
   - $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(a) = 0 < v_{\mathbf{t}}$, for all $\mathbf{t} \in E(G_{\mathbf{r}})$.

   The last two statements bring to $aS_{\mathbf{r}}a$. Being S an arbitrary outranking relation, we obtain that $aS^{N}_{\mathbf{r}}a$ and by point 1. of this Proposition $aS^{P}_{\mathbf{r}}a$; being a an arbitrary alternative, we obtain that $S^{N}_{\mathbf{r}}$ and $S^{P}_{\mathbf{r}}$ are reflexive relations.

3. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and $a, b \in A$ such that $aS^{N}_{\mathbf{r}}b$. This means that for all compatible outranking models, a outranks b with respect to criterion $G_{\mathbf{r}}$; thus there does not exist a compatible outranking model for which a does not outrank b with respect to criterion $G_{\mathbf{r}}$, that is $not(aS^{CP}_{\mathbf{r}}b)$. Conversely, let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and $a, b \in A$ such that $not(aS^{CP}_{\mathbf{r}}b)$. This means that it is not true that there exists one compatible outranking model for which a does not outrank b with respect

to criterion $G_{\mathbf{r}}$. Thus, for all compatible outranking models a outranks b with respect to criterion $G_{\mathbf{r}}$, that is $aS^N_{\mathbf{r}}b$.

4. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and $a,b \in A$ such that $aS^P_{\mathbf{r}}b$. This means that there exists at least one compatible outranking model for which a outranks b with respect to criterion $G_{\mathbf{r}}$; thus, it is not true that a does not outrank b with respect to criterion $G_{\mathbf{r}}$ for all compatible outranking models, that is $not(aS^{CN}_{\mathbf{r}}b)$.
   Conversely, let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and $a,b \in A$ such that $not(aS^{CN}_{\mathbf{r}}b)$. This means that it is not true that for all compatible outranking models a does not outrank b with respect to criterion $G_{\mathbf{r}}$. Thus, there exists at least one compatible outranking model for which a outranks b with respect to criterion $G_{\mathbf{r}}$, that is $aS^P_{\mathbf{r}}b$.

5. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and $a,b \in A$ such that $aS^{CN}_{\mathbf{r}}b$. This means that a does not outrank b for all compatible outranking models; thus there exists at least one compatible outranking model for which a does not outrank b, that is $aS^{CP}_{\mathbf{r}}b$.

6. For all $a \in A$, by points 2. and 3. of this Proposition, we have:
$$aS^N_{\mathbf{r}}a \Leftrightarrow not(aS^{CP}_{\mathbf{r}}a),$$
   and thus $S^{CP}_{\mathbf{r}}$ is an irreflexive binary relation.
   Analogously, for all $a \in A$, by points 2. and 4. of this Proposition, we have:
$$aS^P_{\mathbf{r}}a \Leftrightarrow not(aS^{CN}_{\mathbf{r}}a),$$
   and thus $S^{CN}_{\mathbf{r}}$ is an irreflexive binary relation.

Proof of Proposition 3.2

1. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, and $a,b \in A$ such that $aS^N_{(\mathbf{r},j)}b$, for all $j = 1,\ldots,n(\mathbf{r})$. This means that $aS_{(\mathbf{r},j)}b$ for all $j = 1,\ldots,n(\mathbf{r})$ and for all compatible outranking models. Let $\overline{M}$ one of these compatible outranking models and $\overline{S}$ the outranking relation induced by $\overline{M}$. By point 1. of Proposition 2.1 we obtain $a\overline{S}_{\mathbf{r}}b$. Being $\overline{M}$ an arbitrary compatible outranking model, we have that a outranks b with respect to criterion $G_{\mathbf{r}}$ for all compatible outranking models, and so $aS^N_{\mathbf{r}}b$.

2. Let $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, and $a,b \in A$, such that

α) $aS^{N}_{(\mathbf{r},j)}b$ for all $j=1,\ldots,n(\mathbf{r}), j\neq w$,

β) $aS^{P}_{(\mathbf{r},w)}b$.

Hypothesis β) implies that there exists at least one compatible outranking model $\overline{M}$ inducing the outranking relation $\overline{S}$ such that $a\overline{S}_{(\mathbf{r},w)}b$. But for the hypothesis α) we have also that $a\overline{S}_{(\mathbf{r},j)}b$, for all $j=1,\ldots,n(\mathbf{r})$ and $j\neq w$. Together, these considerations imply that $a\overline{S}_{(\mathbf{r},j)}b$ for all $j=1,\ldots,n(\mathbf{r})$, and thus by point 1. of Proposition 2.1 we obtain that a outranks b with respect to criterion $G_{\mathbf{r}}$ for outranking model $\overline{M}$ and thus $aS^{P}_{\mathbf{r}}b$.

3. Let $G_{\mathbf{r}}\in\mathcal{G}$ with $\mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus\{LBO\cup EL\}$, and $a,b\in A$ such that $aS^{CN}_{(\mathbf{r},j)}b$, for all $j=1,\ldots,n(\mathbf{r})$. This means that $not(aS_{(\mathbf{r},j)}b)$, for all $j=1,\ldots,n(\mathbf{r})$ and for all compatible outranking models. Considering $\overline{M}$ one of these compatible outranking models, and $\overline{S}$ the outranking relation induced by $\overline{M}$, by point 2. of Proposition 2.1 we obtain $not(a\overline{S}_{\mathbf{r}}b)$. Being $\overline{M}$ an arbitrary compatible outranking model, we have $not(aS_{\mathbf{r}}b)$ for all compatible outranking models and thus $aS^{CN}_{\mathbf{r}}b$.

4. Let $G_{\mathbf{r}}\in\mathcal{G}$ with $\mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus\{LBO\cup EL\}$, and $a,b\in A$ such that:

   α) $aS^{CN}_{(\mathbf{r},j)}b$, for all $j=1,\ldots,n(\mathbf{r}), j\neq w$,

   β) $aS^{CP}_{(\mathbf{r},w)}b$,

   β) implies that there exist at least one compatible outranking model $\overline{M}$ inducing the outranking relation $\overline{S}$, such that $not(a\overline{S}_{(\mathbf{r},w)}b)$. By α) we have also that $not(a\overline{S}_{(\mathbf{r},j)}b)$, for all $j=1,\ldots,n(\mathbf{r})$ and $j\neq w$. Together, these considerations imply that $not(a\overline{S}_{(\mathbf{r},j)}b)$ for all $j=1,\ldots,n(\mathbf{r})$, and thus by point 2. of Proposition 2.1 we obtain $not(a\overline{S}_{\mathbf{r}}b)$. Therefore, $aS^{CP}_{\mathbf{r}}b$.

Proof of Proposition 4.1

1. Without loss of generality we have supposed that the hierarchy is structured in a way that each subcriterion at level l descends from only one criterion of level $l-1$ (see Section 2.1). In this way, considering criterion $G_{\mathbf{r}}$ and its subcriteria $G_{(\mathbf{r},1)},\ldots,G_{(\mathbf{r},n(\mathbf{r}))}$ we have:

   - $\pi_{\mathbf{r}}(a,b)=\displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{r}})}k_{\mathbf{t}}P_{\mathbf{t}}(a,b)$,
   - $\pi_{(\mathbf{r},j)}(a,b)=\displaystyle\sum_{\mathbf{t}\in E(G_{(\mathbf{r},j)})}k_{\mathbf{t}}P_{\mathbf{t}}(a,b)$, for all $j=1,\ldots,n(\mathbf{r})$,
   - $E(G_{\mathbf{r}})=\cup_{j=1}^{n(\mathbf{r})}E(G_{(\mathbf{r},j)})$.

We can observe that each $\mathbf{t} \in E(G_{\mathbf{r}})$ belongs to only one of $E(G_{(\mathbf{r},j)})$, $j = 1, \ldots, n$, and thus:

$$\pi_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} k_{\mathbf{t}} P_{\mathbf{t}}(a,b) = \sum_{j=1}^{n(\mathbf{r})} \left[\sum_{\mathbf{t} \in E(G_{(\mathbf{r},j)})} k_{\mathbf{t}} P_{\mathbf{t}}(a,b) \right] = \sum_{j=1}^{n(\mathbf{r})} \pi_{(\mathbf{r},j)}(a,b).$$

2. For each criterion/subcriterion $G_{\mathbf{r}} \in \mathcal{G}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, supposing that there exist n different alternatives in A, we have for all $a \in A$:

   - $\Phi^+_{\mathbf{r}}(a) = \dfrac{1}{n-1} \displaystyle\sum_{x \in A \setminus \{a\}} \pi_{\mathbf{r}}(a,x)$,
   - $\Phi^+_{(\mathbf{r},j)}(a) = \dfrac{1}{n-1} \displaystyle\sum_{x \in A \setminus \{a\}} \pi_{(\mathbf{r},j)}(a,x)$, for all $j = 1, \ldots, n(\mathbf{r})$.

   Thus, by point 1. of this Proposition and using the above expressions:

$$\Phi^+_{\mathbf{r}}(a) = \frac{1}{n-1} \sum_{x \in A \setminus \{a\}} \pi_{\mathbf{r}}(a,x) = \frac{1}{n-1} \sum_{x \in A \setminus \{a\}} \left[\sum_{j=1}^{n(\mathbf{r})} \pi_{(\mathbf{r},j)}(a,x) \right] = \sum_{j=1}^{n(\mathbf{r})} \left[\frac{1}{n-1} \sum_{x \in A \setminus \{a\}} \pi_{(\mathbf{r},j)}(a,x) \right] =$$

$$= \sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(a).$$

3. Analogous to proof of point 2.

4. By points 2. and 3. of this Proposition, for each $a \in A$, and for each $G_{\mathbf{r}} \in \mathcal{G}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$,

$$\Phi_{\mathbf{r}}(a) = \Phi^+_{\mathbf{r}}(a) - \Phi^-_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(a) - \sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} \left[\Phi^+_{(\mathbf{r},j)}(a) - \Phi^-_{(\mathbf{r},j)}(a) \right] =$$

$$= \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(a).$$

Proof of Proposition 4.2

1. Let $a, b \in A$ and $G_{\mathbf{r}} \in \mathcal{G}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that $aP^I_{(\mathbf{r},j)}b$, for all $j = 1, \ldots, n(\mathbf{r})$. By hypothesis, we have:

$$\Phi^+_{(\mathbf{r},j)}(a) \geq \Phi^+_{(\mathbf{r},j)}(b) \quad \text{and} \quad \Phi^-_{(\mathbf{r},j)}(a) \leq \Phi^-_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}),$$

   and for each j at least one of the above inequalities is strict. Then adding up with respect to j, we obtain:

$$\sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(b) \quad \text{and} \quad \sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(a) \leq \sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(b),$$

and thus by points 2. and 3. of Proposition 4.1,

$$\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \quad \text{and} \quad \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b)$$

with at least one of the two inequalities being strict; therefore $aP^I_{\mathbf{r}}b$.

2. Let $a, b \in A$, $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and $\{C_1, C_2\}$ a partition of $\{1, \ldots, n(\mathbf{r})\}$, such that $aP^I_{(\mathbf{r},j)}b$, for all $j \in C_1$ and $aI^I_{(\mathbf{r},j)}b$, for all $j \in C_2$. By the first hypothesis, we have:

$$\Phi^+_{(\mathbf{r},j)}(a) \geq \Phi^+_{(\mathbf{r},j)}(b) \quad \text{and} \quad \Phi^-_{(\mathbf{r},j)}(a) \leq \Phi^-_{(\mathbf{r},j)}(b), \text{ for all } j \in C_1 \tag{3}$$

with at least one of the two inequalities strict; by the second hypothesis we have:

$$\Phi^+_{(\mathbf{r},j)}(a) = \Phi^+_{(\mathbf{r},j)}(b) \quad \text{and} \quad \Phi^-_{(\mathbf{r},j)}(a) = \Phi^-_{(\mathbf{r},j)}(b), \ \text{ for all } j \in C_2. \tag{4}$$

Thus, by points 2. and 3. of Proposition 4.1,

$$\sum_{j \in C_1} \Phi^+_{(\mathbf{r},j)}(a) \geq \sum_{j \in C_1} \Phi^+_{(\mathbf{r},j)}(b) \quad \text{and} \quad \sum_{j \in C_1} \Phi^-_{(\mathbf{r},j)}(a) \leq \sum_{j \in C_1} \Phi^-_{(\mathbf{r},j)}(b)$$

with at least one of the two inequalities strict; by (3) and (4) we obtain:

$$\sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(a) = \sum_{j \in C_1} \Phi^+_{(\mathbf{r},j)}(a) + \sum_{j \in C_2} \Phi^+_{(\mathbf{r},j)}(a) \geq \sum_{j \in C_1} \Phi^+_{(\mathbf{r},j)}(b) + \sum_{j \in C_2} \Phi^+_{(\mathbf{r},j)}(b) = \sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(b)$$

and

$$\sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(a) = \sum_{j \in C_1} \Phi^-_{(\mathbf{r},j)}(a) + \sum_{j \in C_2} \Phi^-_{(\mathbf{r},j)}(a) \leq \sum_{j \in C_1} \Phi^-_{(\mathbf{r},j)}(b) + \sum_{j \in C_2} \Phi^-_{(\mathbf{r},j)}(b) = \sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(b);$$

therefore

$$\sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} \Phi^+_{(\mathbf{r},j)}(b) \quad \text{and} \quad \sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(a) \leq \sum_{j=1}^{n(\mathbf{r})} \Phi^-_{(\mathbf{r},j)}(b),$$

that is, by points 2. and 3. of Proposition 4.1,

$$\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \quad \text{and} \quad \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b)$$

with at least one of the two inequalities being strict. From this follows that $aP^I_{\mathbf{r}}b$.

3. Let $a, b \in A$ and $G_{\mathbf{r}} \in \mathcal{G}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ such that $aI^{I}_{(\mathbf{r},j)}b$, for all $j = 1, \ldots, n(\mathbf{r})$. This means that

$$\Phi^{+}_{(\mathbf{r},j)}(a) = \Phi^{+}_{(\mathbf{r},j)}(b) \quad \text{and} \quad \Phi^{-}_{(\mathbf{r},j)}(a) = \Phi^{-}_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}).$$

Adding up with respect to j we obtain:

$$\sum_{j=1}^{n(\mathbf{r})} \Phi^{+}_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi^{+}_{(\mathbf{r},j)}(b) \quad \text{and} \quad \sum_{j=1}^{n(\mathbf{r})} \Phi^{-}_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi^{-}_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}),$$

that is, by points 2. and 3. of Proposition 4.1,

$$\Phi^{+}_{\mathbf{r}}(a) = \Phi^{+}_{\mathbf{r}}(b) \quad \text{and} \quad \Phi^{-}_{\mathbf{r}}(a) = \Phi^{-}_{\mathbf{r}}(b),$$

and therefore $aI^{I}_{\mathbf{r}}b$.

Proof of Proposition 4.3

1. Let $a, b \in A$, $G_{\mathbf{r}} \in \mathcal{G}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ such that $aP^{II}_{(\mathbf{r},j)}b$, for all $j = 1, \ldots, n(\mathbf{r})$. By point 4. of Proposition 4.1,

$$\Phi_{(\mathbf{r},j)}(a) > \Phi_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}) \Rightarrow \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(a) > \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(b) \Leftrightarrow \Phi_{\mathbf{r}}(a) > \Phi_{\mathbf{r}}(b),$$

and therefore $aP^{II}_{\mathbf{r}}b$.

2. Let $a, b \in A$, $G_{\mathbf{r}} \in \mathcal{G}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ and $\{C_1, C_2\}$ a partition of $\{1, \ldots, n(\mathbf{r})\}$ such that $aP^{II}_{(\mathbf{r},j)}b$ for all $j \in C_1$ and $aI^{II}_{(\mathbf{r},j)}b$ for all $j \in C_2$. By hypothesis we have:

$$\Phi_{(\mathbf{r},j)}(a) > \Phi_{(\mathbf{r},j)}(b), \ \text{ for all } j \in C_1 \quad \text{and} \quad \Phi_{(\mathbf{r},j)}(a) = \Phi_{(\mathbf{r},j)}(b), \ \text{ for all } j \in C_2.$$

Adding up with respect to j, by point 4. of Proposition 4.1, we obtain:

$$\sum_{j \in C_1} \Phi_{(\mathbf{r},j)}(a) > \sum_{j \in C_1} \Phi_{(\mathbf{r},j)}(b) \Rightarrow \sum_{j \in C_1} \Phi_{(\mathbf{r},j)}(a) + \sum_{j \in C_2} \Phi_{(\mathbf{r},j)}(a) > \sum_{j \in C_1} \Phi_{(\mathbf{r},j)}(b) + \sum_{j \in C_2} \Phi_{(\mathbf{r},j)}(b) \Leftrightarrow$$

$$\Leftrightarrow \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(a) > \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(b) \Leftrightarrow \Phi_{\mathbf{r}}(a) > \Phi_{\mathbf{r}}(b),$$

and therefore $aP^{II}_{\mathbf{r}}b$.

3. Let $a, b \in A$, $G_{\mathbf{r}} \in \mathcal{G}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that $aI^{II}_{(\mathbf{r},j)}b$, for all $j = 1, \ldots, n(\mathbf{r})$. By point 4. of Proposition 4.1,

$$\Phi_{(\mathbf{r},j)}(a) = \Phi_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}) \Rightarrow \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} \Phi_{(\mathbf{r},j)}(b) \Leftrightarrow \Phi_{\mathbf{r}}(a) = \Phi_{\mathbf{r}}(b),$$

and therefore $aI^{II}_{\mathbf{r}}b$.

Proof of Proposition 5.1

We prove this Proposition in case of PROMETHEE I because the proof in case of PROMETHEE II is analogous.

1. Let be $a, b \in A$, and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that $a \succsim^N_{\mathbf{r}} b$. This means that a outranks b with respect to criterion $G_{\mathbf{r}}$ for all compatible outranking models; thus, there exists at least one compatible outranking model for which a outranks b with respect to criterion $G_{\mathbf{r}}$, and therefore $a \succsim^P_{\mathbf{r}} b$.

2. For each $a \in A$, for each criterion/subcriterion $G_{\mathbf{r}}$, and for each compatible outranking model, we have:

$$\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(a) \text{ and } \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(a); \tag{5}$$

By equation (5) it follows that, for all compatible outranking models $\Phi_{\mathbf{r}}(a) = \Phi^+_{\mathbf{r}}(a) - \Phi^-_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(a)$ and thus $a \succsim^N_{\mathbf{r}} a$, for all $a \in A$ proving that $\succsim^N_{\mathbf{r}}$ is a reflexive binary relation. Being $\succsim^N_{\mathbf{r}} \subseteq \succsim^P_{\mathbf{r}}$, and $\succsim^N_{\mathbf{r}}$ a reflexive binary relation, also $\succsim^P_{\mathbf{r}}$ is a reflexive binary relation.

Proof of Proposition 5.2

We prove this Proposition in case of PROMETHEE I because the proof in case of PROMETHEE II is analogous.

1. Let $a, b \in A$ and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, such that $a \succsim^N_{(\mathbf{r},j)} b$ for all $j = 1, \ldots, n(\mathbf{r})$. This means that a outranks b with respect to criteria/subcriteria $G_{(\mathbf{r},j)}$, for all $j = 1, \ldots, n(\mathbf{r})$, for all compatible outranking models. Thus, for all compatible outranking models we have:

$$\Phi^+_{(\mathbf{r},j)}(a) \geq \Phi^+_{(\mathbf{r},j)}(b) \text{ and } \Phi^-_{(\mathbf{r},j)}(a) \leq \Phi^-_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}). \tag{6}$$

By points 2. and 3. of Proposition 4.1 and equation above, for all compatible outranking models we have that:

$$\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \text{ and } \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b),$$

implying that $a \succsim_{\mathbf{r}}^{N} b$.

2. Let $a, b \in A$ and $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, such that $a \succsim_{(\mathbf{r},j)}^{N} b$, for all $j = 1, \ldots, n(r), j \neq w$ and $a \succsim_{(\mathbf{r},w)}^{P} b$. This means that a outranks b with respect to criteria $G_{(\mathbf{r},j)}$, for all $j = 1, \ldots, n(\mathbf{r}), j \neq w$ for all compatible outranking models and a outranks b with respect to criterion/subcriterion $G_{(\mathbf{r},w)}$ for at least one compatible outranking model. From this we have that, for all compatible outranking models:

$$\Phi^{+}_{(\mathbf{r},j)}(a) \geq \Phi^{+}_{(\mathbf{r},j)}(b) \text{ and } \Phi^{-}_{(\mathbf{r},j)}(a) \leq \Phi^{-}_{(\mathbf{r},j)}(b), \text{ for all } j = 1, \ldots, n(\mathbf{r}), j \neq w, \tag{7}$$

and for at least one compatible outranking model:

$$\Phi^{+}_{(\mathbf{r},w)}(a) \geq \Phi^{+}_{(\mathbf{r},w)}(b) \text{ and } \Phi^{-}_{(\mathbf{r},w)}(a) \leq \Phi^{-}_{(\mathbf{r},w)}(b). \tag{8}$$

Let us denote by $\overline{M}$ the outranking model satisfying equation (8). In particular, this compatible outranking model fulfills also equation (7). Thus, by points 2. and 3. of Proposition 4.1, and considering the compatible outranking model $\overline{M}$, we have:

$$\Phi^{+}_{\mathbf{r}}(a) \geq \Phi^{+}_{\mathbf{r}}(b) \text{ and } \Phi^{-}_{\mathbf{r}}(a) \leq \Phi^{-}_{\mathbf{r}}(b),$$

and therefore a outranks b with respect to criterion/subcriterion $G_{\mathbf{r}}$ for at least one compatible outranking model, that is $a \succsim_{\mathbf{r}}^{P} b$.

Appendix B

Ordinal regression constraints used in the Hierarchical ELECTRE^{GKMS} method

Supposing that the DM has given some preference information of the type described in section 3.1, compatible outranking models are the sets of variables $\psi_{\mathbf{t}}(a, b)$ for all $(a, b) \in B$, $\mathbf{t} \in EL$, of concordance indices $C_{\mathbf{r}}(a, b)$, concordance cutting levels $\lambda_{\mathbf{s}}$, for all $\mathbf{s} \in LBO$, and veto thresholds $v_{\mathbf{t}}$ for all $\mathbf{t} \in EL$, satisfying the following set of conditions:

- Compatibility with all statements concerning the truth or falsity of the outranking relation for some reference alternatives $a, b \in A^{R}$:

- For all $(a,b) \in B^R$ such that $aS_{\mathbf{r}}b$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$:

$$C_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} \psi_{\mathbf{t}}(a,b) \geq \lambda_{\mathbf{r}} \;\; \text{and} \;\; g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) < v_{\mathbf{t}}, \text{ for all } \mathbf{t} \in E(G_{\mathbf{r}}),$$

- For all $(a,b) \in B^R$ such that $not(aS_{\mathbf{r}}b)$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$:

$$C_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} \psi_{\mathbf{t}}(a,b) < \lambda_{\mathbf{r}} \;\; \text{or} \;\; \text{there exists} \;\; \mathbf{t} \in E(G_{\mathbf{r}}) : g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq v_{\mathbf{t}},$$

  which can be modeled as:

$$C_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} \psi_{\mathbf{t}}(a,b) + \varepsilon \leq \lambda_{\mathbf{r}} + M_0^{\mathbf{r}}(a,b) \;\; \text{and} \;\; g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq v_{\mathbf{t}} - \delta_{\mathbf{r}} M_{\mathbf{t}}(a,b),$$

  where:

  * $M_0^{\mathbf{r}}(a,b), M_{\mathbf{t}}(a,b) \in \{0,1\}$, for all $\mathbf{t} \in E(G_{\mathbf{r}})$,
  * $M_0^{\mathbf{r}}(a,b) + \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} M_{\mathbf{t}}(a,b) \leq |E(G_{\mathbf{r}})|$,
  * $\delta_{\mathbf{r}}$ is an auxiliary coefficient fixed on a big positive value (i.e. $\delta_{\mathbf{r}} \geq max_{\mathbf{t} \in E(G_{\mathbf{r}})}\{\beta_{\mathbf{t}} - \alpha_{\mathbf{t}}\}$ where $\alpha_{\mathbf{t}} = \min_{a \in A} g_{\mathbf{t}}(a)$ and $\beta_{\mathbf{t}} = \max_{a \in A} g_{\mathbf{t}}(a)$).

  Differently from [9], we have one binary variable $M_0^{\mathbf{r}}(a,b)$ for each criterion $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, and for each couple $(a,b) \in B^R$, because we need to distinguish the reasons for which the outranking of alternative a over alternative b is not true. In fact, let us suppose, for example, that alternative a does not outrank alternative b with respect to criteria $G_{\mathbf{r}_1}$ and $G_{\mathbf{r}_2}$, and that in the first case, a does not outrank b because there is an elementary subcriterion descending from $G_{\mathbf{r}_1}$ putting veto while the concordance test is verified. At the same time, let us suppose that a does not outrank b with respect to $G_{\mathbf{r}_2}$ because the concordance test is not verified. Then, in the first case $M_0^{\mathbf{r}_1}(a,b) = 1$, because the concordance test is verified, and in the second case $M_0^{\mathbf{r}_2}(a,b) = 0$, because the concordance test is not verified.

- Constraints on the values of $\lambda_{\mathbf{r}}$, for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, inter-criteria parameters and of $v_{\mathbf{t}}$ and $k_{\mathbf{t}}$, for all $\mathbf{t} \in EL$:

  - Normalization of the marginal concordance indices for all elementary subcriteria, so that the indices corresponding to the greatest difference in evaluations of two alternatives on each elementary subcriterion $(g_{\mathbf{t}}(x^*_{\mathbf{t}}) - g_{\mathbf{t}}(x_{\mathbf{t},*}) = \beta_{\mathbf{t}} - \alpha_{\mathbf{t}})$ sum up to 1:

$$\sum_{\mathbf{t}\in EL} \psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) = 1 \text{ with } x^*_{\mathbf{t}}, x_{\mathbf{t},*} \in A : g_{\mathbf{t}}(x^*_{\mathbf{t}}) = \beta_{\mathbf{t}} \text{ and } g_{\mathbf{t}}(x_{\mathbf{t},*}) = \alpha_{\mathbf{t}}, \text{ for all } \mathbf{t} \in EL.$$

As we normalize weights of the elementary subcriteria so that they sum up to 1, each weight is understood as a maximal share of each elementary subcriterion in the comprehensive concordance index. Consequently, $k_{\mathbf{t}} = \psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*})$, for all $\mathbf{t} \in EL$.

- Lower and upper bounds on concordance cutting level of a criterion belonging to last but one level:

$$\lambda_{\mathbf{s}} \in \left[\frac{K_{\mathbf{s}}}{2}, K_{\mathbf{s}}\right], \text{ where } K_{\mathbf{s}} = \sum_{\mathbf{t}\in E(G_{\mathbf{s}})} k_{\mathbf{t}}.$$

  In consequence of the above considerations, the concordance cutting levels of criteria belonging to the last but one level have to verify:

$$\sum_{\mathbf{t}\in E(G_{\mathbf{s}})} \frac{\psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*})}{2} \leq \lambda_{\mathbf{s}} \leq \sum_{\mathbf{t}\in E(G_{\mathbf{s}})} \psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}).$$

- The concordance cutting level for criterion $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{LBO \cup EL\}$, is equal to the sum of the concordance cutting levels of subcriteria descending from it, that is $G_{(\mathbf{r},\mathbf{j})}, j = 1, \ldots, n(\mathbf{r})$:

$$\lambda_{\mathbf{r}} = \sum_{j=1}^{n(\mathbf{r})} \lambda_{(\mathbf{r},j)}.$$

- Constraints on veto thresholds $v_{\mathbf{t}},\ \mathbf{t} \in EL$,:

  * $v_{\mathbf{t}} > p^*_{\mathbf{t}}$, for each $\mathbf{t} \in EL_1$,
  * $v_{\mathbf{t}} > g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a)$, for each $\mathbf{t} \in EL_2$ such that $a \sim_{\mathbf{t}} b$.

- Constraints on the values of marginal concordance indices $\psi_{\mathbf{t}}(a,b),\ \mathbf{t} \in EL$ conditioned by intra-criterion and inter-criterion preference information, for all $(a,b) \in B$:

  - $k_{\mathbf{t},*} \leq \psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}) \leq k^*_{\mathbf{t}}$, $\mathbf{t} \in EL$,
  - $\psi_{\mathbf{t_1}}(x^*_{\mathbf{t_1}}, x_{\mathbf{t_1},*}) \geq \psi_{\mathbf{t_2}}(x^*_{\mathbf{t_2}}, x_{\mathbf{t_2},*}) + \varepsilon$ if elementary subcriterion $g_{\mathbf{t_1}}$ is more important than elementary subcriterion $g_{\mathbf{t_2}}$, $\mathbf{t_1}, \mathbf{t_2} \in EL$,
  - $\psi_{\mathbf{t_1}}(x^*_{\mathbf{t_1}}, x_{\mathbf{t_1},*}) = \psi_{\mathbf{t_2}}(x^*_{\mathbf{t_2}}, x_{\mathbf{t_2},*})$ if elementary subcriteria $g_{\mathbf{t_1}}$ and $g_{\mathbf{t_2}}$ are equally important, $\mathbf{t_1}, \mathbf{t_2} \in EL$,
  - $\psi_{\mathbf{t}}(a,b) = 0$ if $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) \geq p^*_{\mathbf{t}}$,

– $\psi_{\mathbf{t}}(a,b) > 0$ if $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > -p_{\mathbf{t},*}$,

– $\psi_{\mathbf{t}}(a,b) = \psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*})$ if $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \geq -q_{\mathbf{t},*}$,

– $\psi_{\mathbf{t}}(a,b) < \psi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*})$ if $g_{\mathbf{t}}(b) - g_{\mathbf{t}}(a) > q^*_{\mathbf{t}}$,

– $\psi_{\mathbf{t}}(a,b) = 0$ if $b \succ_{\mathbf{t}} a$,

– $\psi_{\mathbf{t}}(a,b) = 0$ and $\psi_{\mathbf{t}}(b,a) = 0$ if $a \sim_{\mathbf{t}} b$.

- Monotonicity of the functions of marginal concordance indices $\psi_{\mathbf{t}}(a,b)$, $\mathbf{t} \in EL$:

$$\psi_{\mathbf{t}}(a,b) \geq \psi_{\mathbf{t}}(c,d) \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d),$$

$$\psi_{\mathbf{t}}(a,b) = \psi_{\mathbf{t}}(c,d) \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) = g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d),$$

Note that all strict inequalities are transformed into weak inequalities involving an auxiliary variable ε in the set of constraints E^{A^R} in the section 3.1.
For example, the constraint $\psi_{\mathbf{t}}(a,b) > 0$ if $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > -p_{\mathbf{t},*}$, becomes $\psi_{\mathbf{t}}(a,b) \geq \varepsilon$ if $g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > -p_{\mathbf{t},*}$.

Ordinal regression constraints used in the Hierarchical PROMETHEE^{GKS} method

Supposing that the DM has given some preference information of the type described in section 5.1, compatible outranking models are the sets of preference indices $\pi_{\mathbf{t}}(a,b)$ for all $(a,b) \in B$, $\mathbf{t} \in EL$ satisfying the following conditions:

- Compatibility with local and global preference relations provided by the DM with respect to a particular criterion $G_{\mathbf{r}}$ in the hierarchy:

  – for all $a, b \in A^R$, and $G_{\mathbf{r}}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, such that $a \succsim_{\pi_{\mathbf{r}}} b$,

$$\pi_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} \pi_{\mathbf{t}}(a,b) \geq \pi_{\mathbf{r}}(b,a) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} \pi_{\mathbf{t}}(b,a),$$

  Relations $\succ_{\pi_{\mathbf{r}}}$ and $\sim_{\pi_{\mathbf{r}}}$ are translated analogously, using strict inequality and equality, respectively.

  – Considering PROMETHEE I:

    * $\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b)$ and $\Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b)$, if $a \succsim_{\Phi_{\mathbf{r}}} b$,
    * $\left.\begin{array}{l}\Phi^+_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) \text{ and } \Phi^-_{\mathbf{r}}(a) \leq \Phi^-_{\mathbf{r}}(b) \text{ and} \\ \Phi^+_{\mathbf{r}}(a) - \Phi^-_{\mathbf{r}}(a) \geq \Phi^+_{\mathbf{r}}(b) - \Phi^-_{\mathbf{r}}(b) + \varepsilon\end{array}\right\} \text{ if } a \succ_{\Phi_{\mathbf{r}}} b$

* $\Phi^{+}_{\mathbf{r}}(a) = \Phi^{+}_{\mathbf{r}}(b)$ and $\Phi^{-}_{\mathbf{r}}(a) = \Phi^{-}_{\mathbf{r}}(b)$ if $a \sim_{\Phi_{\mathbf{r}}} b$.

– Considering PROMETHEE II:

$$\Phi_{\mathbf{r}}(a) = \Phi^{+}_{\mathbf{r}}(a) - \Phi^{-}_{\mathbf{r}}(a) \geq \Phi_{\mathbf{r}}(b) = \Phi^{+}_{\mathbf{r}}(b) - \Phi^{-}_{\mathbf{r}}(b) \text{ if } a \succsim_{\Phi_{\mathbf{r}}} b.$$

Relations $\succ_{\Phi_{\mathbf{r}}}$, and $\sim_{\Phi_{\mathbf{r}}}$ are treated analogously, using strict inequality and equality, respectively.

- Normalization of the marginal preference indices for all criteria, so that the indices corresponding to the greatest difference in evaluations of two alternatives on each elementary subcriterion $(g_{\mathbf{t}}(x^{*}_{\mathbf{t}}) - g_{\mathbf{t}}(x_{\mathbf{t},*}) = \beta_{\mathbf{t}} - \alpha_{\mathbf{t}})$ sum up to 1:

$$\sum_{\mathbf{t}\in EL} \pi_{\mathbf{t}}(x^{*}_{\mathbf{t}}, x_{\mathbf{t},*}) = 1 \text{ with } x^{*}_{\mathbf{t}}, x_{\mathbf{t},*} \in A, \text{ for all } \mathbf{t} \in EL.$$

  We normalize weights of the criteria, so that they sum up to 1. Therefore, each weight is now understood as a maximal share of each elementary subcriterion in the aggregated preference index. Consequently, $k_{\mathbf{t}} = \pi_{\mathbf{t}}(x^{*}_{\mathbf{t}}, x_{\mathbf{t},*})$, for all $\mathbf{t} \in EL$.

- Restrictions concerning the value of marginal preference indices $\pi_{\mathbf{t}}, \mathbf{t} \in EL$:

  – $\pi_{\mathbf{t}}(a,b)$ needs to be equal 0 if a is not better than b on elementary subcriterion $g_{\mathbf{t}}$ by more than the least value of an indifference threshold $q_{\mathbf{t},*}$ allowed by the DM:

$$\pi_{\mathbf{t}}(a,b) = 0 \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \leq q_{\mathbf{t},*}, \text{ for all } (a,b) \in B, \mathbf{t} \in EL_1;$$

  – $\pi_{\mathbf{t}}(a,b)$ needs to be greater than 0 if a is better than b on elementary subcriterion $g_{\mathbf{t}}$ by more than the greatest value of an indifference threshold $q^{*}_{\mathbf{t}}$ allowed by the DM:

$$\pi_{\mathbf{t}}(a,b) > 0 \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > q^{*}_{\mathbf{t}}, \text{ for all } (a,b) \in B, \mathbf{t} \in EL_1;$$

  – $\pi_{\mathbf{t}}(a,b)$ needs to be less than the maximal value of the preference index on elementary subcriterion $g_{\mathbf{t}}$ if a is not better than b by more than the least value of a preference threshold $p_{\mathbf{t},*}$ allowed by the DM;

$$\pi_{\mathbf{t}}(a,b) < \pi_{\mathbf{t}}(x^{*}_{\mathbf{t}}, x_{\mathbf{t},*}), \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) < p_{\mathbf{t},*}, \text{ for all } (a,b) \in B, \mathbf{t} \in EL_1;$$

  – $\pi_{\mathbf{t}}(a,b)$ needs to be equal to the maximal value of preference index on elementary subcriterion $g_{\mathbf{t}}$

if a is better than b by more than the greatest value of a preference threshold $p^*_{\mathbf{t}}$ allowed by the DM:

$$\pi_{\mathbf{t}}(a,b) = \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}), \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) \geq p^*_{\mathbf{t}}, \text{ for all } (a,b) \in B, \mathbf{t} \in EL_1;$$

– $\pi_{\mathbf{t}}(a,b)$ and $\pi_{\mathbf{t}}(b,a)$ need to be equal to 0 if the difference between $g_{\mathbf{t}}(a)$ and $g_{\mathbf{t}}(b)$ is not-significant for the DM:

$$\pi_{\mathbf{t}}(a,b) = 0, \ \pi_{\mathbf{t}}(b,a) = 0 \text{ if } a \sim_{\mathbf{t}} b, \mathbf{t} \in EL_2;$$

– $\pi_{\mathbf{t}}(a,b)$ needs to be equal to the maximal value of the preference index on criterion $g_{\mathbf{t}}$ if the difference between $g_{\mathbf{t}}(a)$ and $g_{\mathbf{t}}(b)$ is significant for the DM:

$$\pi_{\mathbf{t}}(a,b) = \pi_{\mathbf{t}}(x^*_{\mathbf{t}}, x_{\mathbf{t},*}), \text{ if } a \succ_{\mathbf{t}} b, \mathbf{t} \in EL_2.$$

- Monotonicity of the functions of marginal preference indices $\pi_{\mathbf{t}}(a,b)$, for all $\mathbf{t} \in EL$:

$$\pi_{\mathbf{t}}(a,b) \geq \pi_{\mathbf{t}}(c,d) \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) > g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d),$$

$$\pi_{\mathbf{t}}(a,b) = \pi_{\mathbf{t}}(c,d) \text{ if } g_{\mathbf{t}}(a) - g_{\mathbf{t}}(b) = g_{\mathbf{t}}(c) - g_{\mathbf{t}}(d).$$

Note that all strict inequalities are transformed into weak inequalities involving an auxiliary variable ε in the set of constraints E^{A^R} in the section 5.1.