Multiple Criteria Hierarchy Process for Sorting Problems Based on Ordinal Regression with Additive Value Functions

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Abstract

A hierarchical decomposition is a common approach for coping with complex decision problems involving multiple dimensions. Recently, the Multiple Criteria Hierarchy Process (MCHP) has been introduced as a new general framework for dealing with multiple criteria decision aiding (MCDA) in case of a hierarchical structure of the family of evaluation criteria. This study applies the MCHP framework to multiple criteria sorting problems and extends existing disaggregation and robust ordinal regression techniques that induce decision models from data. The new methodology allows the handling of preference information and the formulation of recommendations at the comprehensive level, as well as at all intermediate levels of the hierarchy of criteria. A case study on bank performance rating is used to illustrate the proposed methodology.

Keywords: Multiple Criteria Decision Aiding, Multiple Criteria Hierarchy Process, Sorting problems, Robust ordinal regression, Bank rating

1 Introduction

In many decision making problems, decisions concerning a set of alternatives are based on different evaluation criteria organized in a hierarchical structure. Such a hierarchy introduces a decomposition of the primary objective into separate dimensions, which are then further analyzed in sub-dimensions, up to the lowest level of the hierarchy, which consists of the elementary criteria. Structuring decision problems following such a hierarchical scheme is particularly useful in situations that require consideration of large sets of criteria describing different aspects of the problem at hand. Dealing with complex families of criteria of diverse nature, poses significant cognitive burden to decision makers (DMs). Thus, using a hierarchical decomposition facilitates the analysis as it allows DMs to deal with more manageable elementary dimensions. Furthermore, working with such a hierarchy provides detailed insights on all partial dimensions of the problem, instead of focusing solely on the comprehensive level.

A common approach to deal with hierarchies of criteria in MCDA is the analytic hierarchy process [25], but its fundamental problems are well-documented in the literature (see, for example, [2]). Recently, the Multiple Criteria Hierarchy Process (MCHP) has been introduced as an alternative [1, 7, 8]. The MCHP introduces a new modeling framework that allows the construction of sound decision models in decision problems with a hierarchical structure, through MCDA techniques based on the preference disaggregation paradigm [21]. The MCHP is able to take into account preference information not only at a comprehensive level but also at all lower levels of the hierarchy, and provide recommendations in a similar form.

In previous studies, the MCHP has been introduced in the context of choice and ranking problems, where the objective is either to choose the best alternative(s) among those considered (choice) or to rank-order the alternatives from the best to the worst ones. In these contexts, the MCHP has been employed to construct decision models with outranking methods such as ELECTRE and PROMETHEE [8], value function models [7], as well as with the Choquet integral preference model [1].

In this study, we extend the MCHP framework to multiple criteria sorting (classification) problems, where the objective is to assign a set of alternatives to predefined (ordinal) decision classes. Such problems often arise in many domains [33] and they have attracted much interest in MCDA over

the past decade. Sorting problems have been dealt in the literature using outranking relations (e.g., the ELECTRE Tri method [31]), value functions, and decision rules [15, 16, 17, 27]. In this paper, we focus on value function models, which constitute a convenient and easy way of modeling DMs preferences in MCDA problems. The best-known method based on this modeling approach for multiple criteria sorting problems is the UTADIS method (UTilités Additives DIScriminantes) and its variants [9, 10, 32]. In this paper, we extend the UTADIS method to problems having a hierarchical structure by applying the MCHP framework. In order to reduce the cognitive effort of the DM, we also extend the UTADIS^{GMS} method [20], putting it in the MCHP framework. UTADIS^{GMS} is the generalization of UTADIS to the Robust Ordinal Regression (ROR) setting [5, 6, 19]. ROR is a family of methods taking into account not only one but all instances of an assumed preference model being compatible with the preference information provided by the DM. In that regard, this study contributes to the literature on multiple criteria sorting through the extension of existing techniques for inferring decision models from sorting decision examples, using a formal framework of MCHP, which allows the input preference information to be decomposed into smaller and more manageable aspects of the problem. In order to illustrate the proposed methodology, we employ a case study involving a financial decision problem, namely the performance rating of banks. In a supervisory context, bank rating is a complex process that requires the consideration of all aspects of bank operation, financial status, and risk profile. This case study fits well the framework of MCHP and multiple criteria sorting, and thus, it illustrates well the potentials of the proposed modeling approach in practice.

The rest of the paper is organized as follows: In the next section, a general problem setting is provided. Section 3 describes the MCHP extension of the UTADIS method to decision problems with a hierarchical structure, while in section 4 the integration of MCHP and UTADIS^{GMS} is explained in detail. The application to bank performance evaluation is presented in section 5. Finally, section 6 concludes the paper and provides some future research directions.

2 General Setting

A set of alternatives $A = \{a, b, \ldots\}$ is evaluated on a set of criteria structured in a hierarchical way in l different levels. The complete set of criteria (from all levels) will be denoted by $\mathcal{G}$, while the set of indices of criteria will be denoted by $\mathcal{I}_{\mathcal{G}}$. The criteria at the lowest level of the hierarchy will be called elementary criteria and the alternatives will be directly evaluated on these criteria only. The set of indices of elementary criteria will be denoted by EL, while the set of indices of elementary criteria

descending from node $G_{\mathbf{r}}$ of the hierarchy ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$), will be denoted by $E(G_{\mathbf{r}})$. Each node of the hierarchy represents a particular sub-dimension of the problem, with $G_{\mathbf{0}}$ corresponding to the root of the hierarchy (i.e., $G_{\mathbf{0}} = \mathcal{G}$). Without loss of generality, we shall suppose that all elementary criteria are to be maximized (i.e., preference increases with the value of each criterion).

Furthermore, by $n(\mathbf{r})$ we shall denote the number of criteria $G_{(\mathbf{r},1)}, \ldots, G_{(\mathbf{r},n(\mathbf{r}))}$ descending from $G_{\mathbf{r}}$ in the next (lower) level of the hierarchy. Obviously, the elementary criteria are not further decomposed into subcriteria. By LBO we shall denote the indices of the criteria from the next to the last level of the hierarchy, while $LB(G_{\mathbf{r}})$ will denote the set of indices for criteria descending from $G_{\mathbf{r}}$ and located at the next to the last level.

Assuming that the set of elementary criteria is mutually preferentially independent [23, 30], their aggregation is possible with an additive value function $U : A \to [0, 1]$, such that:

$$U(a) = \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(g_{\mathbf{t}}(a))$$

where $u_{\mathbf{t}}$ are marginal value functions related to elementary criteria $g_{\mathbf{t}}$.

In the MCHP context, assuming that at each level criteria are preferentially independent, it is possible to consider a partial value function for each (non-elementary) criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$ as follows:

$$U_{\mathbf{r}}(a) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(g_{\mathbf{t}}(a)).$$

An obvious consequence is that

$$U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) \tag{1}$$

where $U_{(\mathbf{r},j)}(a)$ represents the value of alternative a according to the j-th subcriterion of $G_{\mathbf{r}}$, for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ (more details on MCHP can be found in [7]).

For each criterion $G_{\mathbf{r}}$ above the level of elementary criteria, the sorting procedure with respect to subcriteria descending directly from $G_{\mathbf{r}}$ consists in assigning each alternative from A to one among $p_{\mathbf{r}}$ decision classes $C_1, \ldots, C_{p_{\mathbf{r}}}$, where $C_{p_{\mathbf{r}}}$ is the class of top performing alternatives and C_1 is the class of the worst alternatives. Note that the sorting with respect to subcriteria descending directly from different criteria $G_{\mathbf{r}}$ could involve different values of $p_{\mathbf{r}}$, i.e., the number of classes to which an alternative can be assigned could depend on $G_{\mathbf{r}}$. For each criterion $G_{\mathbf{r}}$ above the elementary level, class C_h ($h \in \{1, \ldots, p_{\mathbf{r}}\}$) is defined by lower and upper value thresholds $b_{h-1}^{\mathbf{r}}$ and $b_h^{\mathbf{r}}$, such that $b_{h-1}^{\mathbf{r}} < b_h^{\mathbf{r}}$,

defined on the value function scale. It follows that $0 = b_0^{\mathbf{r}} < b_1^{\mathbf{r}} < \ldots < b_{p_{\mathbf{r}}-1}^{\mathbf{r}} < b_{p_{\mathbf{r}}}^{\mathbf{r}} = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}})$, where the value of $b_{p_{\mathbf{r}}}^{\mathbf{r}}$ is the maximum level of the value function for $G_{\mathbf{r}}$ (with $x_{\mathbf{t}}^{m_{\mathbf{t}}}$ being the best performance on elementary criterion $g_{\mathbf{t}}$ over all alternatives from A).

3 MCHP and the UTADIS method

Consider an assignment of alternative $a \in A$ to class C_h ($h \in \{1, ..., p_{\mathbf{r}}\}$) with respect to subcriteria descending directly from criterion $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$). In the following, instead of criterion $G_{\mathbf{r}}$, we shall often use the term *node* $G_{\mathbf{r}}$, in order to stress that the assignment takes place in a particular place of the hierarchy tree.

Moreover, the assignment of alternatives with respect to subcriteria descending directly from criterion $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$) will be called the *assignment in node* $G_{\mathbf{r}}$.

Definition 3.1. *In node* $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$), *alternative* a *is assigned to class* C_h ($h = 1, \ldots, p_{\mathbf{r}}$) $\left(\text{denoted as } a \underset{\mathbf{r}}{\rightarrow} C_h\right)$, *iff* $b_{h-1}^{\mathbf{r}} \leq U_{\mathbf{r}}(a) < b_h^{\mathbf{r}}$.

As a consequence, in node $G_{\mathbf{r}}$,

- a *is assigned to at least class* C_h $\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$, *iff* $U_{\mathbf{r}}(a) \geq b_{h-1}^{\mathbf{r}}$,
- a *is assigned to at most class* C_h $\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$, *iff* $U_{\mathbf{r}}(a) < b_h^{\mathbf{r}}$ *(the inequality becomes weak if* $h = p_{\mathbf{r}}$, *that is* $U_{\mathbf{r}}(a) \leq b_{p_{\mathbf{r}}}^{\mathbf{r}}$*),*
- a *is assigned to some class in the interval* $[C_{h_1}, C_{h_2}]$ *(*$1 < h_1 < h_2 < p_{\mathbf{r}}$*)* $\left(a \underset{\mathbf{r}}{\rightarrow} [C_{h_1}, C_{h_2}]\right)$, *iff* $b_{h_1-1}^{\mathbf{r}} \leq U_{\mathbf{r}}(a) < b_{h_2}^{\mathbf{r}}$.

In what follows, in order to simplify the presentation and without loss of generality, we assume that the same classes apply in all nodes of the hierarchy tree. This means that the number of classes $p_{\mathbf{r}}$ to which each alternative can be assigned does not depend on the considered node $G_{\mathbf{r}}$. Consequently, $p_{\mathbf{r}} = p$ for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$.

A first desirable coherence property for hierarchical multiple criteria sorting methods is the following. If an alternative $a \in A$ is assigned to class C_h in all nodes directly descending from $G_{\mathbf{r}}$, then it should also be assigned to the same class in node $G_{\mathbf{r}}$. For example, if student S is assigned to the class of good students in the nodes corresponding to Algebra and Analysis, being the only two subcriteria of the criterion Mathematics, then S has to be assigned to the class of good students also in the node of Mathematics. Henceforth, we shall refer to this property as the *first coherence property of hierarchical multiple criteria sorting.*

A second desirable coherence property for hierarchical multiple criteria sorting methods is the following. If an alternative $a \in A$ is assigned to at least class C_h, i.e., to class C_h or better, in all nodes directly descending from $G_{\mathbf{r}}$, then it should also be assigned to at least class C_h in node $G_{\mathbf{r}}$. Coming back to the previous example, if student S is assigned to at least medium class of students in the nodes corresponding to Algebra and Analysis, then S has also to be assigned to at least the medium class of students in the node of Mathematics. Let us call this property *second coherence property of hierarchical multiple criteria sorting*. Of course, another coherence property for hierarchical multiple criteria sorting methods is symmetric to the second property, i.e., if an alternative $a \in A$ is assigned to at most class C_h (to class C_h or worse), in all nodes directly descending from $G_{\mathbf{r}}$, then it should also be assigned to at most class C_h in node $G_{\mathbf{r}}$. Henceforth, this property will be referred to as the *third coherence property of hierarchical multiple criteria sorting*. The second and third coherence properties of hierarchical multiple criteria sorting can be synthesized as follows: in node $G_{\mathbf{r}}$, an alternative $a \in A$ should be assigned to an interval of contiguous classes, included in the interval of classes having as extrema the worst and the best classes to which a is assigned in nodes directly descending from $G_{\mathbf{r}}$. For example, if student S is assigned to the interval of classes from moderate to relatively good students in the node of Algebra, and to the interval of classes from medium to good students in the node of Analysis, then student S has to be assigned to an interval of classes from moderate to good students in the node of Mathematics. Even if this coherence property is the mere synthesis of the above second and third coherence properties, we shall refer to it as the *fourth coherence property for hierarchical multiple criteria sorting methods*, because it will be useful to recall it in the subsequent discussion.

Proposition 3.1 given below says that the first and the fourth coherence properties for hierarchical multiple criteria sorting methods coincide, and that they hold if and only if the value thresholds separating the classes in node $G_{\mathbf{r}}$ are equal to the sum of the corresponding value thresholds separating the classes in the nodes directly descending from $G_{\mathbf{r}}$ (see the Appendix for the proofs). Indeed, this condition is expressed as statement 1 of Proposition 3.1, whereas the first and the fourth coherence properties for hierarchical multiple criteria sorting methods correspond to statements 2 and 3, respectively.

Proposition 3.1. *The three following statements are equivalent:*

1. *In each node* $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $b_h^{\mathbf{r}} = \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ *for all* $h = 0, \ldots, p$,
2. *In each node* $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, *if* $a \xrightarrow[(\mathbf{r},j)]{} [C_{h_j}, C_{k_j}]$ *for all* $j = 1, \ldots, n(\mathbf{r})$, *then* $a \underset{\mathbf{r}}{\rightarrow} [C_h, C_k]$

where $h = \min_{j=1,\ldots,n(\mathbf{r})} h_j$, *and* $k = \max_{j=1,\ldots,n(\mathbf{r})} k_j$,

3. *In each node* $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, *if* $a \xrightarrow[(\mathbf{r},j)]{} C_h$ *for all* $j = 1, \ldots, n(\mathbf{r})$ *then* $a \underset{\mathbf{r}}{\rightarrow} C_h$.

Since we would like our hierarchical sorting approach to respect points 2 and 3 of Proposition 3.1, we shall assume that $b_h^{\mathbf{r}} = \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ in each node $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$. As a consequence of this choice, it is sufficient to define the value thresholds in nodes from the last but one level of the hierarchy, because for any other higher level node $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL \cup LBO\}$ it holds that

$$b_j^{\mathbf{r}} = \sum_{\mathbf{s} \in LB(G_{\mathbf{r}})} b_j^{\mathbf{s}}, \text{ for all } j = 0, \ldots, p.$$

In order to construct the additive value function and define the value thresholds, one can use a direct or an indirect approach. In the former case, the DM is asked to specify explicitly the parameters of the model (value thresholds in this case), following an direct assessment protocol designed specifically for the type of model under consideration (for example of such a protocol for additive value models see [3]). On the other hand, in an indirect approach [21] the DM is asked to provide some comprehensive preference information on the assignment of some reference alternatives (i.e., taking into account the full set of criteria present in the hierarchy) and/or partial preference information (i.e., considering a particular dimension of the problem, corresponding to criterion $G_{\mathbf{r}}$, being an intermediate node in the hierarchy tree). With such preference information at hand, it is possible to infer values for the parameters of the model that are compatible with the judgments provided by the DM. This can be achieved considering the following set of constraints (in accordance with [20], henceforth (U, b) will be used to denote a value function and a set of value thresholds compatible with the preferences of the DM, whereas $\mathcal{U}$ will denote the set of all compatible instances of this model):

$$
\left.
\begin{array}{l}
\left.
\begin{array}{l}
U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_{h-1}, \\
U_{\mathbf{r}}(a) - b^{\mathbf{r}}_{h} \leq -\varepsilon
\end{array}
\right\} \text{if } a \underset{\mathbf{r}}{\rightarrow} C_h \\
U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_{h-1}, \text{ if } a \underset{\mathbf{r}}{\rightarrow} C_{\geq h} \\
U_{\mathbf{r}}(a) - b^{\mathbf{r}}_{h} \leq -\varepsilon, \text{ if } a \underset{\mathbf{r}}{\rightarrow} C_{\leq h} \\
\left.
\begin{array}{l}
U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_{h_1-1}, \\
U_{\mathbf{r}}(a) - b^{\mathbf{r}}_{h_2} \leq -\varepsilon
\end{array}
\right\} \text{if } a \underset{\mathbf{r}}{\rightarrow} [C_{h_1}, C_{h_2}] \\
u_{\mathbf{t}}(x^k_{\mathbf{t}}) \geq u_{\mathbf{t}}(x^{k-1}_{\mathbf{t}}), \quad k = 1, \ldots, m_{\mathbf{t}}, \text{ for all } \mathbf{t} \in EL, \\
u_{\mathbf{t}}(x^0_{\mathbf{t}}) = 0, \text{ for all } \mathbf{t} \in EL, \text{ and } \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}) = 1 \\
b^{\mathbf{s}}_h \geq b^{\mathbf{s}}_{h-1} + \varepsilon, h = 1, \ldots, p, \text{ for all } \mathbf{s} \in LBO, \\
b^{\mathbf{s}}_0 = 0, \text{ and } b^{\mathbf{s}}_p = \sum_{\mathbf{t} \in E(G_{\mathbf{s}})} u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}), \text{ for all } \mathbf{s} \in LBO, \\
b^{\mathbf{r}}_h = \sum_{\mathbf{s} \in LB(G_{\mathbf{r}})} b^{\mathbf{s}}_h, \text{for all } h = 0, \ldots, p, \text{ and for all } \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL \cup LBO\}
\end{array}
\right\} E^{A^R}
$$

where $x^k_{\mathbf{t}}$, $k = 0, \ldots, m_{\mathbf{t}}$, are the $m_{\mathbf{t}} + 1$ different performances on elementary criterion $g_{\mathbf{t}}$ attained by alternatives in A (arranged in ascending order); $x^0_{\mathbf{t}}$ and $x^{m_{\mathbf{t}}}_{\mathbf{t}}$ are, respectively, the worst and the best performances of alternatives on elementary criterion $g_{\mathbf{t}}$, while ε is an auxiliary variable used to translate the strict inequality constraints to weak inequality constraints.

If E^{A^R} is feasible and $\varepsilon^* > 0$, where $\varepsilon^* = \max \varepsilon$ subject to E^{A^R}, then there exists at least one instance (U, b) compatible with the preferences provided by the DM. The readers interested to conditions ensuring the existence of an additive representation of ordered partitions could look at [3].

Remark 3.1. *Let us observe that if the number of classes considered in each node $G_{\mathbf{r}}$ is different (different values of $p_{\mathbf{r}}$ for all $G_{\mathbf{r}}$), then the direct and the indirect approaches explained above remain valid. In particular, in the indirect approach, one has to consider the set of constraints $E_1^{A^R}$ obtained from E^{A^R} by replacing the last three constraints with the following ones:*

$$
b^{\mathbf{r}}_h \geq b^{\mathbf{r}}_{h-1} + \varepsilon, h = 1, \ldots, p_{\mathbf{r}}, \quad \textit{for all } \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL, \tag{2}
$$

$$
b^{\mathbf{r}}_0 = 0, \textit{ and } b^{\mathbf{r}}_{p_{\mathbf{r}}} = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}), \quad \textit{for all } \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL. \tag{3}
$$

The last constraint in E^{A^R} does not hold anymore (except for the case $h = 0$ and $h = p_{\mathbf{r}}$ in consequence

of eq. (3)), since the number and meaning of the different thresholds obviously depend on the number of classes to which each alternative can be assigned in node $G_{\mathbf{r}}$. For example, let us consider a small hierarchy in which a root criterion $G_{\mathbf{0}}$ has subcriteria $G_{\mathbf{1}}$ and $G_{\mathbf{2}}$, and each alternative can be assigned to two classes in node $G_{\mathbf{0}}$ ($p_{\mathbf{0}} = 2$) while it can be assigned to three and four classes in nodes $G_{\mathbf{1}}$ and $G_{\mathbf{2}}$, respectively ($p_{\mathbf{1}} = 3$ and $p_{\mathbf{2}} = 4$). In this case, there will be three value thresholds in node $G_{\mathbf{0}}$ $\left(\left\{b_0^0, b_1^0, b_2^0\right\}\right)$, four value thresholds in node $G_{\mathbf{1}}$ $\left(\left\{b_0^1, b_1^1, b_2^1, b_3^1\right\}\right)$, and five value thresholds in node $G_{\mathbf{2}}$ $\left(\left\{b_0^2, b_1^2, b_2^2, b_3^2, b_4^2\right\}\right)$.

4 MCHP and the UTADIS^{GMS} method

In general, more than one instance of the preference model could be compatible with the preference information provided by the DM. Each of these instances restores the given information in the same way, but each one of them could provide different recommendations on alternatives outside the reference set. In this case, choosing a single compatible instance of the preference model may lead to a loss of possibly important information. For this reason, Robust Ordinal Regression (ROR) [5, 6, 19] takes into account the whole set of instances of the preference model compatible with the preference information provided by the DM, by building *necessary* and *possible* preference relations that hold for all or for at least one compatible instance of the preference model.

In the MCHP context, the DM could be therefore interested to know not only to which class an alternative could be necessarily or possibly assigned taking into account the whole set of criteria, but also to which class it could be necessarily or possibly assigned with respect to a criterion corresponding to a particular node of the hierarchy tree.

In this section, we extend UTADIS^{GMS} [20] to the MCHP context, by reformulating the definition of the necessary and possible assignments as follows:

Definition 4.1. *In any node $G_{\mathbf{r}}$, ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$) in the hierarchy tree,*

- *$a \in A$ is necessarily assigned to at least class C_h, denoted by $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$, iff $U_{\mathbf{r}}(a) \geq b_{h-1}^{\mathbf{r}}$ for all compatible (U, b),*
- *$a \in A$ is possibly assigned to at least class C_h, denoted by $a \xrightarrow[\mathbf{r}]{P} C_{\geq h}$, iff $U_{\mathbf{r}}(a) \geq b_{h-1}^{\mathbf{r}}$ for at least one compatible (U, b),*
- *$a \in A$ is necessarily assigned to at most class C_h, denoted by $a \xrightarrow[\mathbf{r}]{N} C_{\leq h}$, iff $U_{\mathbf{r}}(a) < b_h^{\mathbf{r}}$ for all compatible (U, b),*

- *$a \in A$ is possibly assigned to at most class C_h, denoted by $a \xrightarrow[\mathbf{r}]{P} C_{\leq h}$, iff $U_{\mathbf{r}}(a) < b_h^{\mathbf{r}}$ for at least one compatible (U, b).*

The above necessary and possible preference relations can be computed as follows:

- $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ iff the set of constraints $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$ is infeasible or if $\varepsilon\left(a \xrightarrow[\mathbf{r}]{N} C_{\geq h}\right) \leq 0$, where $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right) = E^{A^R} \cup \left\{U_{\mathbf{r}}(a) + \varepsilon \leq b_{h-1}^{\mathbf{r}}\right\}$ and $\varepsilon\left(a \xrightarrow[\mathbf{r}]{N} C_{\geq h}\right) = \max \varepsilon$, s.t. $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$;
- $a \xrightarrow[\mathbf{r}]{P} C_{\geq h}$ iff the set of constraints $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$ is feasible and $\varepsilon\left(a \xrightarrow[\mathbf{r}]{P} C_{\geq h}\right) > 0$, where $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right) = E^{A^R} \cup \left\{U_{\mathbf{r}}(a) \geq b_{h-1}^{\mathbf{r}}\right\}$ and $\varepsilon\left(a \xrightarrow[\mathbf{r}]{P} C_{\geq h}\right) = \max \varepsilon$, s.t. $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$;
- $a \xrightarrow[\mathbf{r}]{N} C_{\leq h}$ iff the set of constraints $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$ is infeasible or if $\varepsilon\left(a \xrightarrow[\mathbf{r}]{N} C_{\leq h}\right) \leq 0$, where $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right) = E^{A^R} \cup \left\{U_{\mathbf{r}}(a) \geq b_h^{\mathbf{r}}\right\}$ and $\varepsilon\left(a \xrightarrow[\mathbf{r}]{N} C_{\leq h}\right) = \max \varepsilon$, s.t. $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$;
- $a \xrightarrow[\mathbf{r}]{P} C_{\leq h}$ iff the set of constraints $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$ is feasible and $\varepsilon\left(a \xrightarrow[\mathbf{r}]{P} C_{\leq h}\right) > 0$, where $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right) = E^{A^R} \cup \left\{U_{\mathbf{r}}(a) + \varepsilon \leq b_h^{\mathbf{r}}\right\}$ and $\varepsilon\left(a \xrightarrow[\mathbf{r}]{P} C_{\leq h}\right) = \max \varepsilon$, s.t. $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$.

Robust hierarchical multiple criteria sorting methods should satisfy some desirable properties. The first two properties are logical properties, that, in fact, have to be satisfied even when there is no hierarchical structure. These properties state that for all $a \in A$ and for any non-elementary criterion $G_{\mathbf{r}}$,

1R) either a is necessarily assigned to at least class C_h, or a is possibly assigned to at most class $C_{h-1}, h \in \{2, \ldots, p\}$,

2R) either a is necessarily assigned to at most class C_k, or a is possibly assigned to at least class $C_{k+1}, k \in \{1, \ldots, p-1\}$.

Conditions **1R)** and **2R)** can be considered as completeness properties for robust hierarchical multiple criteria sorting corresponding to completeness properties for "flat" (non-hierarchical) sorting problems, according to which for all $a \in A$:

1B) either a is assigned to at least class C_h, or a is assigned to at most class $C_{h-1}, h \in \{2, \ldots, p\}$,

2B) either a is assigned to at most class C_k, or a is assigned to at least class $C_{k+1}, k \in \{1, \ldots, p-1\}$.

On the basis of this observation, we shall call properties **1R)** and **2R)**, *first and second completeness properties of robust hierarchical multiple criteria sorting.* Observe that removing the reference to node $G_{\mathbf{r}}$ of the hierarchy tree, the first and the second completeness properties should hold for any robust multiple criteria sorting method.

Other desirable properties are related to the hierarchical nature of the robust sorting, and can be seen as counterparts of the robust multiple criteria sorting of the coherence properties considered in section 3 for non-hierarchical multiple criteria sorting. The coherence properties for robust hierarchical multiple criteria sorting methods that we shall consider are the following:

- If a is necessarily assigned to at least class C_h in all nodes directly descending from $G_{\mathbf{r}}$, then it is necessarily assigned to at least class C_h in node $G_{\mathbf{r}}$. For example, if student S is necessarily assigned to at least class medium in both Algebra and Analysis, then S has to be assigned to at least class medium also in the node of Mathematics. Let us call this property *first coherence property for robust hierarchical multiple criteria sorting methods.*

- If a is necessarily assigned to at most class C_k in all nodes directly descending from $G_{\mathbf{r}}$, then it is necessarily assigned to at most class C_k in node $G_{\mathbf{r}}$. For example, if student S is necessarily assigned to at most class moderate in both Algebra and Analysis, then S has to be assigned to at most class moderate also in the node of Mathematics. Let us call this property *second coherence property for robust hierarchical multiple criteria sorting methods.*

- If a is necessarily assigned to at least class C_h in all nodes directly descending from $G_{\mathbf{r}}$, with the possible exception of node $\overline{j}$ for which a is possibly assigned to at least class C_h, then a is possibly assigned to at least class C_h in node $G_{\mathbf{r}}$. For example, if student S is assigned to at least class medium necessarily in the node of Algebra, and possibly in the node of Analysis, then S has to be possibly assigned to at least class medium also in the node of Mathematics. Let us call this property *third coherence property for robust hierarchical multiple criteria sorting methods.*

- If a is necessarily assigned to at most class C_k in all nodes directly descending from $G_{\mathbf{r}}$, with the possible exception of node $\overline{j}$ for which a is possibly assigned to at most class C_k, then a is possibly assigned to at most class C_k in node $G_{\mathbf{r}}$. For example, if student S is assigned to at most class moderate necessarily in the node of Algebra, and possibly in the node of Analysis, then S has to be possibly assigned to at most class moderate also in the node of Mathematics. Let us call this property *fourth coherence property for robust hierarchical multiple criteria sorting methods.*

Proposition 4.1 given below says that the above two completeness properties, as well as the four coherence properties hold for the hierarchical UTADIS^{GMS} we are proposing. Notice that the four

coherence properties are satisfied because statement 1 in Proposition 3.1 holds, i.e., because the value thresholds separating the classes in node $G_{\mathbf{r}}$ are equal to the sum of corresponding value thresholds separating the classes in all nodes directly descending from $G_{\mathbf{r}}$.

Proposition 4.1. *In any node $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, of the hierarchy tree,*

1. *For all $a \in A$, and $h = 2, \ldots, p$, either $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ or $a \xrightarrow[\mathbf{r}]{P} C_{\leq h-1}$,*
2. *For all $a \in A$, and $k = 1, \ldots, p-1$, either $a \xrightarrow[\mathbf{r}]{N} C_{\leq k}$ or $a \xrightarrow[\mathbf{r}]{P} C_{\geq k+1}$,*
3. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}, j = 1, \ldots, n(\mathbf{r})$, then $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ where $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$,*
4. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}, j = 1, \ldots, n(\mathbf{r})$, then $a \xrightarrow[\mathbf{r}]{N} C_{\leq k}$ where $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$,*
5. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}, j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\bar{j}\}$ and $a \xrightarrow[(\mathbf{r},\bar{j})]{P} C_{\geq h_{\bar{j}}}$, then $a \xrightarrow[\mathbf{r}]{P} C_{\geq h}$ where $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$,*
6. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}, j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\bar{j}\}$, and $a \xrightarrow[(\mathbf{r},\bar{j})]{P} C_{\leq k_{\bar{j}}}$ then $a \xrightarrow[\mathbf{r}]{P} C_{\leq k}$ where $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$.*

An obvious consequence of Proposition 4.1 is that if $a \xrightarrow[(\mathbf{r},j)]{N} C_h, j = 1, \ldots, n(\mathbf{r})$, then $a \xrightarrow[\mathbf{r}]{N} C_h$.

In order to possibly or necessarily assign an alternative $a \in A$ to an interval of classes in node $G_{\mathbf{r}}$ of the hierarchy tree, the following indices can be defined:

$$L_{\mathbf{r}}^{\mathcal{U},P}(a) = \max\left(\{1\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{N} C_{\geq h}\right\}\right), \quad R_{\mathbf{r}}^{\mathcal{U},P}(a) = \min\left(\{p\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{N} C_{\leq h}\right\}\right) \quad (4)$$

$$L_{\mathbf{r}}^{\mathcal{U},N}(a) = \max\left(\{1\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{P} C_{\geq h}\right\}\right), \quad R_{\mathbf{r}}^{\mathcal{U},N}(a) = \min\left(\{p\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{P} C_{\leq h}\right\}\right). \quad (5)$$

On the basis of Proposition 4.1, we can prove the following results:

Proposition 4.2. *In any node $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$) of the hierarchy tree, and $a \in A$,*

1. $L_{\mathbf{r}}^{\mathcal{U},P}(a) \geq \min\limits_{j=1,\ldots,n(\mathbf{r})} \left\{L_{(\mathbf{r},j)}^{\mathcal{U},P}(a)\right\}$,
2. $R_{\mathbf{r}}^{\mathcal{U},P}(a) \leq \max\limits_{j=1,\ldots,n(\mathbf{r})} \left\{R_{(\mathbf{r},j)}^{\mathcal{U},P}(a)\right\}$.

5 Application to bank performance rating

5.1 Problem context

In order to illustrate the applicability of the proposed approaches, this section presents results from a case study involving bank performance rating, in a context of prudential supervision. Under the existing financial regulatory framework of Basel II, the banking supervisory authorities of each country (e.g. central banks) should conduct performance assessments on a regular basis for banks operating in the country, in order to ensure the stability of the country's banking system. Given that bank defaults are rare events, adequate historical data are usually not available to fit statistical models for estimating the likelihood of financial distress for banking institutions. Therefore, supervisors mainly rely on judgmental peer assessment systems, which take into account all aspects of a bank's operations and risk profile [24, 26, 29]. The application of the MCDA is well-suited in this context, as it provides bank analysts and supervisors with a formal framework and analytic techniques for constructing composite performance indicators, exploring the trade-offs between different risk and performance factors, conducting robustness checks, and exploring stress testing scenarios.

Typically, bank rating systems consider six major dimensions, which define a comprehensive assessment framework referred to as CAMELS:

1) capital adequacy,

2) asset quality,

3) management competence,

4) earning generating ability,

5) liquidity,

6) sensitivity to market risks.

These dimensions are further decomposed into elementary criteria, which are specified according to particular characteristics of the banking system in a country. Thus, the problem has a hierarchical structure and the bank rating assessment process should provide results not only at the comprehensive level, but also at each one of the above main dimensions. The results are commonly expressed in a 5-point rating scale. Thus, the context of bank rating fits well the MCHP sorting framework developed in this study.

5.2 Data and criteria

The data used in the analysis are taken from [11] and they originate from the Bank of Greece (the supervisory authority responsible for the Greek banking system). They involve 18 Greek banks between 2001 and 2005 (overall 85 bank-year observations[1], which correspond to the alternatives). The banks have been evaluated on 31 criteria structured in a hierarchical way following the CAMELS framework, as shown in Figure 1. The six CAMELS dimensions (Capital-CA, Assets-AS, Management-MC, Earnings-ER, Liquidity-LQ, and Sensitivity to market risks-SM) are the first level criteria, each analyzed through multiple subcriteria in the subsequent level. These subcriteria serve as the elementary decision attributes in the MCHP framework, for which the data are available for the banks in the sample.

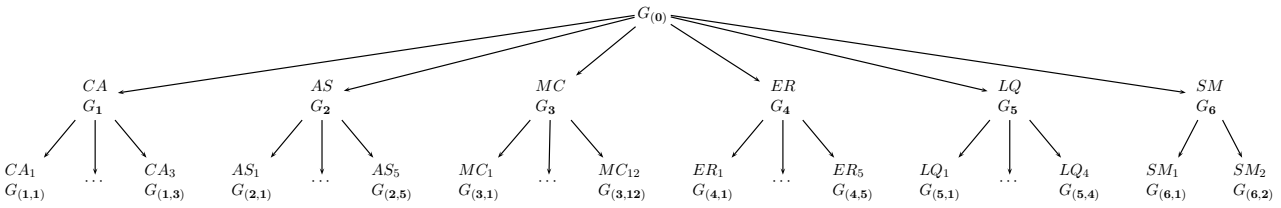

Figure 1: Hierarchy of Criteria

The definition of the elementary criteria is given in Table 1. These include 17 financial ratios that describe quantitative aspects of bank operation, whereas the remaining 14 criteria describe qualitative issues (but these are still measured on a $0.5 - 5.5$ cardinal scale defined by analysts at the Bank of Greece, with lower values indicating higher performance). Criteria whose type is indicated in Table 1 as "max" are positively related to the performance of banks, whereas minimization criteria are those that are negatively related to bank performance.

For each elementary criterion $g_{\mathbf{t}}$, we considered a linear marginal value function:

$$u_{\mathbf{t}}(a) = u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \frac{g_{\mathbf{t}}(a) - x_{\mathbf{t}}^{0}}{x_{\mathbf{t}}^{m_{\mathbf{t}}} - x_{\mathbf{t}}^{0}} \tag{6}$$

where the best ($x_{\mathbf{t}}^{m_{\mathbf{t}}}$) and worst ($x_{\mathbf{t}}^{0}$) performances are defined as follows:

$$\begin{aligned} &\text{Maximization criteria:} \quad x_{\mathbf{t}}^{m_{\mathbf{t}}} = \max\{g_{\mathbf{t}}(a), a \in A\} \quad \text{and} \quad x_{\mathbf{t}}^{0} = \min\{g_{\mathbf{t}}(a), a \in A\} \\ &\text{Minimization criteria:} \quad x_{\mathbf{t}}^{m_{\mathbf{t}}} = \min\{g_{\mathbf{t}}(a), a \in A\} \quad \text{and} \quad x_{\mathbf{t}}^{0} = \max\{g_{\mathbf{t}}(a), a \in A\}. \end{aligned}$$

[1]For some banks the data were not available for all years. The data are available at http://www.fel.tuc.gr/BankData.xlsx

Table 1: Evaluation criteria and their indices used in Figure 1

Category	Index	Abbr.	Type	Index	Criterion name
Capital	**1**	CA1	Max	$(\mathbf{1},1)$	Capital adequacy ratio
		CA2	Min	$(\mathbf{1},2)$	TIER II capital / TIER I
		CA3	Min	$(\mathbf{1},3)$	Qualitative*
Assets	**2**	AS1	Min	$(\mathbf{2},1)$	Risk-weighted assets / Assets
		AS2	Min	$(\mathbf{2},2)$	(Non performing loans – Provisions) / Loans
		AS3	Min	$(\mathbf{2},3)$	Large exposures / (TIER I + TIER II capital)
		AS4	Min	$(\mathbf{2},4)$	[0.5(Non performing loans) – Provisions]/Equity
		AS5	Min	$(\mathbf{2},5)$	Qualitative*
Management	**3**	MC1	Min	$(\mathbf{3},1)$	Operating expenses / Operating income
		MC2	Min	$(\mathbf{3},2)$	Staff cost / Assets
		MC3	Max	$(\mathbf{3},3)$	Operating income / Business units
		MC4	Min	$(\mathbf{3},4)$	Top management competencies, qualifications and continuity
		MC5	Min	$(\mathbf{3},5)$	Managers' experience and competence
		MC6	Min	$(\mathbf{3},6)$	Resilience to change, strategy, long term horizon
		MC7	Min	$(\mathbf{3},7)$	Management of information systems
		MC8	Min	$(\mathbf{3},8)$	Internal control systems
		MC9	Min	$(\mathbf{3},9)$	Financial risk management system
		MC10	Min	$(\mathbf{3},10)$	Internal processes charter - implementation monitoring
		MC11	Min	$(\mathbf{3},11)$	Timely and accurate data collection
		MC12	Min	$(\mathbf{3},12)$	Information technology systems
Earnings	**4**	ER1	Max	$(\mathbf{4},1)$	Net income / Assets
		ER2	Max	$(\mathbf{4},2)$	Net income / Equity
		ER3	Max	$(\mathbf{4},3)$	Interest revenue / Assets
		ER4	Max	$(\mathbf{4},4)$	Other operating revenue / Assets
		ER5	Min	$(\mathbf{4},5)$	Qualitative*
Liquidity	**5**	LQ1	Max	$(\mathbf{5},1)$	Cash / Assets
		LQ2	Min	$(\mathbf{5},2)$	(Loans – Provisions) / Deposits
		LQ3	Min	$(\mathbf{5},3)$	Real funding from credit institutions / Assets
		LQ4	Min	$(\mathbf{5},4)$	Qualitative*
Market	**6**	SM1	Min	$(\mathbf{6},1)$	Risk-weighted assets II / Risk-weighted Assets (I & II)
		SM2	Min	$(\mathbf{6},2)$	Qualitative*

* Undisclosed criteria related to qualitative aspects of the banks' operation

It should be noted that the use of linear marginal value functions in the setting of this case study, is actually in accordance with the CAMELS modeling framework as implemented by the Bank of Greece. Furthermore, similar linear scoring and risk monitoring systems are widely used by bank supervisory agencies worldwide.

In accordance with the policy followed by analysts at the Bank of Greece during the period under consideration, the following points are taken into consideration:

- The importance of quantitative criteria should be at least equal to 70%. Even though criteria related to qualitative aspects of bank operation are particularly useful for describing important performance and risk factors in the medium-long term, they clearly entail some subjectivity

on the way they are modeled and assessed. On the other hand, financial quantitative criteria, despite their shortcomings (e.g., potential manipulation of accounting reporting standards), are hard data widely used in prudential supervision research and practice all over the world. In that regard, this requirement is imposed to ensure that the resulting evaluation does not overweight the qualitative aspects of bank operation over the actual financial results.

Denoting by $\mathcal{I}_{\mathcal{G}_{Qual}}$ the set of indices of qualitative elementary criteria, that is

$$\mathcal{I}_{\mathcal{G}_{Qual}} = \{(\mathbf{1}, 3), (\mathbf{2}, 5), (\mathbf{4}, 5), (\mathbf{5}, 4), (\mathbf{6}, 2)\}$$

and by $\mathcal{I}_{\mathcal{G}_{Quan}}$ the set of indices of all quantitative elementary criteria ($\mathcal{I}_{\mathcal{G}_{Qual}} \cup \mathcal{I}_{\mathcal{G}_{Quan}} = \mathcal{I}_{\mathcal{G}}$), the previous piece of preference information can be translated to the following constraint:

$$\sum_{\mathbf{t} \in \mathcal{I}_{\mathcal{G}_{Quan}}} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq 0.7. \tag{7}$$

This implies that a bank having the best performance on all quantitative elementary criteria should have a comprehensive value not less than 0.7.

- Capital and assets are the most important dimensions, whereas market risk is the least important one. Capital adequacy and asset quality are critical factors for ensuring the financial soundness of a bank. They are both closely monitored on a regular basis by supervisors, and actions are taken whenever a bank does not have adequate capital (see for example the stress tests conducted by the European Banking Authority) or when its loan portfolio is particularly troublesome. Liquidity is also an important issue, but during the period of the analysis (2001–2005) there were no indications that liquidity risk could be a critical factor in the foreseeable future for Greek banks. Therefore, liquidity is considered to be of lower importance for this analysis, compared to capital adequacy and asset quality. The same applies to earning power and management competence, too. Earning power is an important dimension for the success of banking institutions as it indicates how they perform in multiple areas. Furthermore, a strong stream of earnings constitutes the first line of defense against loan losses. However, the period of the analysis was a time of transition for Greek banks in terms of their profitability, mainly due to the introduction of the Euro and the adoption of the international accounting standards by the largest banks. Due to the challenges that these issues created in assessing the earnings of Greek

banking institutions over the period under consideration, its relative importance was set below capital and assets. On the other hand, management competence is mostly related to qualitative aspects of bank operation, which, as explained above, are given lower priority. Finally, the data set only involves commercial banks, whose exposure to market risks is limited. Therefore, the market risk dimension is assumed to be the least important one among the six criteria categories.

Using the notation introduced in section 2 and indices of criteria shown in Table 1, the given three pieces of preference information can be translated to the following sets of constraints

$$\sum_{\mathbf{t}\in E(G_1)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq \begin{cases} \sum_{\mathbf{t}\in E(G_3)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_4)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_5)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_6)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \end{cases} \qquad \sum_{\mathbf{t}\in E(G_2)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq \begin{cases} \sum_{\mathbf{t}\in E(G_3)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_4)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_5)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_6)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \end{cases} \tag{8}$$

$$\sum_{\mathbf{t}\in E(G_6)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \leq \begin{cases} \sum_{\mathbf{t}\in E(G_3)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) - \varepsilon, \\ \sum_{\mathbf{t}\in E(G_4)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) - \varepsilon, \\ \sum_{\mathbf{t}\in E(G_5)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) - \varepsilon, \end{cases} \tag{9}$$

where the constraints (8) say that criteria categories capital and assets are more important than the other four criteria categories, while the constraints (9) say that market risk is the least important criteria category. Let us notice that the constraints saying that market risk is less important than capital and assets are missing in (9) since these constraints are already present in (8).

5.3 Discussion of results

In addition to the above preference information, an expert banking analyst (DM) familiar with the Greek banking sector provided global assessments for a small set of banks, as shown in Table 2. These are banks for which the DM was familiar with their strengths and weaknesses over the examined period. For example, alternatives A_3, A_4, and A_5 correspond to a leading Greek bank in terms of its market niche and financial strength over a three years period (2003–2005), alternatives A_7, A_8, and

A_9 involve a state-owned bank being in transition towards privatization, whereas A_{16}, A_{17}, and A_{18} correspond to a recently privatized bank that faced significant operating challenges moving to a new corporate plan.

Table 2: Initial set of the expert's comprehensive judgments

Alternatives	Class
A_{16}, A_{17}, A_{18}	C_1
A_{10}, A_{21}, A_{22}	C_2
A_7, A_8, A_9	C_3
A_1, A_2, A_6	C_4
A_3, A_4, A_5	C_5

Since each bank can be assigned to one of five classes at the level of macro-criteria and at the comprehensive level, then six thresholds have to be specified for each macro-criterion (b_0^s, b_1^s, b_2^s, b_3^s, b_4^s, b_5^s), such that $b_0^s = 0$ and $b_5^s = \sum_{\mathbf{t}\in E(G_s)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}})$ for all $s \in \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}\}$. Consequently, following Proposition 3.1, the thresholds for criterion G_0 are obtained as the sum of the corresponding thresholds for the six macro-criteria, that is $b_h^0 = \sum_{s=1}^{6} b_h^s$, for all $h = 0,\ldots,5$.
Having defined the thresholds for the six macro-criteria, the preferences shown in Table 2 are translated to constraints as explained in section 3. For example, the assignment at a comprehensive level of bank A_x to class C_h is translated to the constraints

$$\left.\begin{aligned} &U_0(A_x) \geq b_{h-1}^0, \\ &U_0(A_x) - b_h^0 \leq -\varepsilon. \end{aligned}\right\} \tag{10}$$

Consequently, the set E^{A^R} containing the constraints translating the preferences of the DM and the technical constraints will be the following:

$$\left.\begin{aligned} &(7)-(10), \\ &u_{\mathbf{t}}(x_{\mathbf{t}}^0) = 0, \text{ for all } \mathbf{t} \in EL, \text{ and } \sum_{\mathbf{t}\in EL} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) = 1 \\ &u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq u_{\mathbf{t}}(x_{\mathbf{t}}^0), \text{ for all } \mathbf{t} \in EL\,, \\ &b_h^{\mathbf{s}} \geq b_{h-1}^{\mathbf{s}} + \varepsilon,\ h = 1,\ldots,5, \text{ for all } \mathbf{s} \in \{\mathbf{1},\ldots,\mathbf{6}\}\,, \\ &b_0^{\mathbf{s}} = 0, \text{ and } b_5^{\mathbf{s}} = \sum_{\mathbf{t}\in E(G_{\mathbf{s}})} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}), \text{ for all } \mathbf{s} \in \{\mathbf{1},\ldots,\mathbf{6}\}\,, \\ &b_h^{\mathbf{0}} = \sum_{\mathbf{s}\in\{\mathbf{1},\ldots,\mathbf{6}\}} b_h^{\mathbf{s}}, \text{for all } h = 0,\ldots,5. \end{aligned}\right\}$$

Note that in this case we do not need the monotonicity constraint $u_{\mathbf{t}}(x_{\mathbf{t}}^k) \geq u_{\mathbf{t}}(x_{\mathbf{t}}^{k-1})$, $k = 1, \ldots, m_{\mathbf{t}}$, for any $\mathbf{t} \in EL$ because, as shown in equation (6), we are considering a linear marginal value function for each elementary criterion, and this function is defined by the marginal value $u_{\mathbf{t}}(x_{\mathbf{t}})$ and by the worst and the best performances of the banks on each elementary criterion.

Solving the LP problem $\varepsilon^* = \max \varepsilon$, s.t. E^{A^R}, we find that E^{A^R} is feasible and $\varepsilon^* > 0$. This leads to the conclusion that there are multiple different instances of the preference model compatible with the above comprehensive judgments and preferential inputs. Clearly, the choice of a single decision instance from such limited information is likely to lead to conclusions that are not robust. Combining ROR with the modeling framework of the UTADIS method under the hierarchical structuring of the family of criteria, enables the formulation of results taking into account the full set of possible instances.

Applying (4), we computed the lowest and the highest possible class assignment for each alternative. Apart from the seven banks shown in Table 3, all the others could be possibly assigned to the whole range of classes. Moreover, applying (5), we computed the lowest and highest necessary assignment for each alternative. It appears that the set of necessary assignments is empty for all banks, since $L_{\mathbf{0}}^{\mathcal{U},N} > R_{\mathbf{0}}^{\mathcal{U},N}$ for all of them. It is evident that at this stage of the analysis, the obtained results are not conclusive enough.

Table 3: Results after the first stage

Alternatives	$\left[L_{\mathbf{0}}^{\mathcal{U},P}, R_{\mathbf{0}}^{\mathcal{U},P}\right]$
A_{23}, A_{54}, A_{61}, A_{68}	$[C_1, C_4]$
A_{36}, A_{80}, A_{81}	$[C_2, C_5]$

In order to get a more clear recommendation, the expert analyst has to provide more detailed preference information. Then, the DM provided partial judgments involving the main CAMELS dimensions, as shown in Table 4. These partial judgments are easier for the DM to define, as each main dimension comprises a much smaller set of criteria compared to the 31 criteria required for the comprehensive assignment decisions provided in the previous stage. The calculation of the new recommendation is performed analogously to the first stage.

With the new preference information, the integration of MCHP with the UTADIS^{GMS} method was employed again to get a new set of assignments. Table 5 reports the number of non-reference cases (i.e., banks-year observations not included in the assignments provided by the expert analyst), by the type of their assignment result (range of classes) at the comprehensive level and at all lower-level

Table 4: Information provided by the expert in the second stage

Capital adequacy		Asset quality		Management competence	
Alternative	Assignment	Alternative	Assignment	Alternative	Assignment
A_{67}	$[C_1, C_2]$	A_{60}	$[C_1, C_2]$	A_{17}	C_2
A_{19}	$[C_2, C_3]$	A_{41}	$[C_2, C_3]$	A_{60}	C_3
A_7	C_3	A_{11}	$[C_3, C_4]$	A_1	C_4
A_1	C_4	A_5	$[C_4, C_5]$		
A_4	C_5				
Earning power		Liquidity		Market risks	
A_{19}	C_1	A_{82}	C_1	A_{76}	$[C_1, C_2]$
A_{20}	C_2	A_{28}	C_2	A_{33}	C_3
A_3	C_3	A_{79}	C_3	A_{22}	C_4
A_{26}	C_4	A_{78}	$[C_3, C_4]$	A_{47}	$[C_4, C_5]$
A_{36}	C_5	A_{55}	C_5		

dimensions. In addition, the table also presents the mean range of the assignments as an indicator of the imprecision that describes the obtained results. The mean range is calculated from the number of classes in the sets of possible assignments, averaged over all non-reference bank-year observations.

Table 5: Summary of possible assignments from the second stage of the analysis (non-reference alternatives)

Assignments	Overall	CA	AS	MC	ER	LQ	SM
C_1	–	–	1	–	–	–	–
$[C_1, C_2]$	–	–	1	–	5	2	2
$[C_1, C_3]$	–	–	–	4	10	26	5
$[C_1, C_4]$	11	–	2	22	9	8	11
$[C_1, C_5]$	40	–	56	18	28	25	–
C_2	–	2	–	–	–	–	–
$[C_2, C_3]$	–	13	–	–	–	–	–
$[C_2, C_4]$	–	2	–	7	–	–	–
$[C_2, C_5]$	16	28	17	24	15	19	–
$[C_3, C_5]$	3	34	4	7	9	–	33
$[C_4, C_5]$	–	3	–	–	3	–	30
C_5	–	–	–	–	1	–	–
Mean range	4.5	3.1	4.6	4.0	3.9	3.9	2.7

It is evident that even with the new information, the sorting decisions at the comprehensive level are still characterized by ambiguity, as 40 (out of 70) cases can be assigned in any of the five rating classes. The examination of the partial assignments for each of the six main dimensions provides some insights on the decomposition of the banks' comprehensive performance and the sources of ambiguity in the assignments at the comprehensive level.

In particular, the partial assignments for capital adequacy (CA) and sensitivity to market risks (SM) are more precise compared to the other dimensions. In terms of capital adequacy, all banks are consistently rated in class C_2 or better (throughout the years), with 37 cases being in at least medium condition (i.e., belonging to categories $C_3 - C_5$). This result is concordant with the characteristics of Greek banks during the period of the analysis, as prior to the outbreak of the Greek sovereign debt crisis in 2010, they have been generally well capitalized.

As far as their sensitivity to market risks is concerned, the banks also performed rather well over the period under consideration. In particular, 63 cases are considered as having at least medium performance on this dimension. There are, however, a few cases corresponding to banks that seem to be exposed to market risks (i.e., their assignment includes the high risk class C_1). These are mostly smaller banks, which have indeed developed some risky investment activities and financial products during that period.

Asset quality seems to be the main factor explaining the ambiguity in the assignment at the comprehensive level. In the majority of cases (56 out of 81), the assignments in this dimension span all five rating classes, which indicates that in order to obtain more precise conclusions on the asset quality dimension, further analysis is required using additional input information. The same applies (yet to a smaller extend) to management competence, earnings, and liquidity.

A further examination of the time trends in the range of the assignments over time (Figure 2) reveals that the imprecision in the assignments at the comprehensive level has increased over time. This can be interpreted as a warning signal, as it implies that deriving clear conclusions on the overall performance of the banks became more difficult over the years. This trend was primarily driven by the increasing ambiguity in the evaluations with respect to capital adequacy (after 2002), asset quality (mostly in 2001–2002), and management competence. On the other hand, the imprecision in the evaluations with respect to the market risk dimension followed a declining trend, as the introduction of Greece to the Eurozone area in 2002 and the improving conditions in the global financial markets (particularly after 2003) contributed to the minimization of the exposure of Greek banks to external market risks.

The information derived from the imprecise assignments of the UTADIS^{GMS} method can be further enriched and complemented through the construction of the most discriminant additive value model, which is obtained through the solution of the optimization problem: $\max \varepsilon$, subject to E^{A^R}. Table 6 presents the number of assignments with the obtained model, both at the comprehensive level and at the level of the six performance dimensions. According to the results, there are six cases

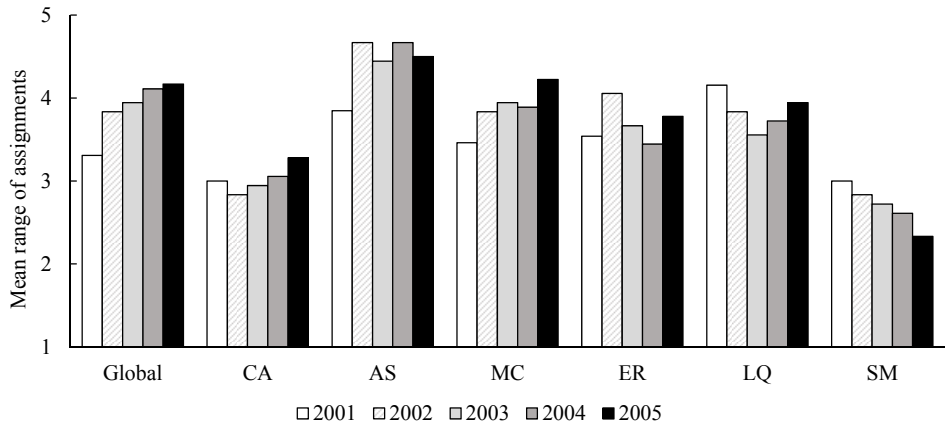

Figure 2: Mean ranges of the assignments over time

Table 6: Summary of assignment results with the most discriminant value function (number of assignments by each class)

Class	Comprehensively	CA	AS	MC	ER	LQ	SM
C_1	6	0	26	21	8	20	6
C_2	23	22	9	29	24	38	4
C_3	19	17	7	30	29	15	15
C_4	32	43	7	5	15	8	42
C_5	5	3	36	0	9	4	18

involving very high risk banks (class C_1), five cases of top performing banks (class C_5), whereas most banks are assigned to classes $C_2 - C_5$. The distribution of the assignments for the capital adequacy dimension resembles the assignment at the comprehensive level, whereas in terms of asset quality it is interesting to note that there is a considerable concentration in the two extreme rating classes. This is in accordance with the large number of imprecise assignments in this dimension, as discussed earlier. In terms of management competence and liquidity there is a concentration in classes $C_1 - C_3$ (at most medium performance), whereas the results for market risk verify the remarks made earlier on the low exposure of Greek banks to external market risks as there is a clear concentration in classes $C_4 - C_5$ (above average performance). The Kendall's τ rank correlations between the comprehensive assignment and the partial ones were higher for capital adequacy (0.725) and asset quality (0.653), which is concordant with the information that the expert analyst provided on the high importance of these criteria. The correlations of the comprehensive assignment to those of the other dimensions were lower (0.2−0.3).

Table 7 presents further results on the relationship of the imprecise assignments obtained by UTADIS^{GMS} with the ones of the most discriminant model at the comprehensive level. In particular,

for banks assigned to different ranges of classes according to UTADIS^{GMS}, we report their mean global values (i.e., performance scores) according to the most representative model (second column), as well as their distribution in the classes resulting from the most discriminant model (frequencies). For instance, the mean performance score for banks assigned to the range of classes $[C_1, C_4]$ is 0.4, and most of such instances (90.9%) are assigned to class C_2 by the most discriminant model. The last row in the table presents the mean comprehensive value for banks assigned to different classes by the most discriminant model. The results indicate that most banks assigned to $[C_1, C_4]$ by UTADIS^{GMS} are considered as low performance banks by the most discriminant model. Banks for which their assignment is completely imprecise according to UTADIS^{GMS} span the whole range of classes with the most discriminant model, but most of them are assigned to the medium performance class C_3. On the other hand, banks assigned to the range of classes $[C_2, C_5]$ and $[C_3, C_5]$ according to UTADIS^{GMS} are assigned to class C_4 by the most discriminant model. However, the mean value of banks in $[C_2, C_5]$ is 0.567, which is very similar to the mean performance (0.559) of banks assigned to C_4 by the most discriminant model (i.e., they resemble typically good banks), whereas banks assigned to $[C_3, C_5]$ have a mean performance value of 0.603, which is higher that the mean of class C_4 but lower that the mean of the top rating class C_5 (0.642).

Table 7: The relationship between the results of UTADIS^{GMS} and the assignments of the most discriminant model at the comprehensive level

		Most discriminant assignments				
	Mean value	C_1	C_2	C_3	C_4	C_5
$[C_1, C_4]$	0.400	9.1%	90.9%	–	–	–
$[C_1, C_5]$	0.483	5.0%	25.0%	40.0%	27.5%	2.5%
$[C_2, C_5]$	0.567	–	–	–	93.8%	6.2%
$[C_3, C_5]$	0.603	–	–	–	100.0%	–
	Mean value	0.337	0.423	0.482	0.559	0.642

6 Conclusions

Several methods are able to deal with multiple criteria sorting problems, but they all assume a single-level organization of the family of criteria. In this paper, we proposed an extension of the MCHP approach to sorting problems with a hierarchical structure of the family of criteria. The MCHP is a methodology that allows the decomposing of decision making problems into smaller dimensions (each taking into account different aspects of the problem). In this context, we introduced modeling

formulations that allow the inference of a preference model from decision examples through preference disaggregation techniques based on an additive value function model (UTADIS and UTADIS^{GMS} methods). MCHP combined with UTADIS and UTADIS^{GMS} allows the consideration of both global and partial preference judgments, which adds flexibility to the specification of the input preference information required in the decision aiding process. The applicability of the MCHP-based methods was illustrated through an application regarding the assessment of bank performance.

Future research can be extended towards a number of different directions. First, similar approaches could be considered for other types of preference models for sorting problems, including outranking relation [4], Choquet integral [13], and rule-based models [18, 28]. That would be particularly useful, as it would yield a much more general MCHP framework, covering situations where different aspects of a decision problem require the adoption of different types of models. Group decision making problems can also be considered in such a context. Combinations with simulation methods [22] could also be useful to enhance the assignment recommendations with probabilistic information, whereas further analysis could also focus on building good representative preference models in sorting problems with hierarchical structure, using the techniques presented in previous studies [12, 14]. In addition to these methodological extensions, further testing on other case studies and through experimental computational analyses could provide further insights into the properties of the MCHP-based sorting schemes. Introduction of procedures guiding the elicitation of preference information by the DM in the spirit of active learning would also be useful to reduce the cognitive effort required during the decision aiding process and make such techniques easier to apply in practice.

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Appendix

A Proof of Proposition 3.1

Proof. (1) $\Rightarrow$ (2) Let $a \xrightarrow[(\mathbf{r},j)]{} \left[C_{h_j}, C_{k_j}\right]$ for all $j = 1, \ldots, n(\mathbf{r})$. This means that $b_{h_j-1}^{(\mathbf{r},j)} \leq U_{(\mathbf{r},j)}(a) < b_{k_j}^{(\mathbf{r},j)}$ for all $j = 1, \ldots, n(\mathbf{r})$. Let us consider $h = \min_{j=1,\ldots,n(\mathbf{r})} h_j$ and $k = \max_{j=1,\ldots,n(\mathbf{r})} k_j$. For the monotonicity

of the thresholds, we shall have for all $j = 1, \ldots, n(\mathbf{r})$ that $b_{h-1}^{(\mathbf{r},j)} \leq b_{h_j-1}^{(\mathbf{r},j)} \leq U_{(\mathbf{r},j)}(a) < b_{k_j}^{(\mathbf{r},j)} \leq b_k^{(\mathbf{r},j)}$ for all $j = 1, \ldots, n(\mathbf{r})$ and, consequently, $b_{h-1}^{(\mathbf{r},j)} \leq U_{(\mathbf{r},j)}(a) < b_k^{(\mathbf{r},j)}$. Adding up with respect to j, we get

$$b_{h-1}^{\mathbf{r}} = \sum_{j=1}^{n(\mathbf{r})} b_{h-1}^{(\mathbf{r},j)} \ \leq \ \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) \ < \ \sum_{j=1}^{n(\mathbf{r})} b_k^{(\mathbf{r},j)} = b_k^{\mathbf{r}}.$$

From equation (1), it follows that $b_{h-1}^{\mathbf{r}} \leq U_{\mathbf{r}}(a) < b_k^{\mathbf{r}}$ and, consequently, $a \underset{\mathbf{r}}{\rightarrow} [C_h, C_k]$.

(2) $\Rightarrow$ (3) follows directly by setting $h_j = k_j = h$ for all $j = 1, \ldots, n(\mathbf{r})$.

(3) $\Rightarrow$ (1) follows by contradiction, when we suppose that $b_h^{\mathbf{r}} \neq \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ for some h. This implies that $b_h^{\mathbf{r}} > \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ or $b_h^{\mathbf{r}} < \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$.

Let $b_h^{\mathbf{r}} > \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ and $a \in A$ an alternative, such that

$$U_{(\mathbf{r},j)}(a) = b_h^{(\mathbf{r},j)} \ \text{ for all } \ j = 1, \ldots, n(\mathbf{r}). \tag{11}$$

Obviously, this implies that $a \underset{(\mathbf{r},j)}{\longrightarrow} C_{h+1}$ for all $j = 1, \ldots, n(\mathbf{r})$. Adding up with respect to j in the two members of equation (11), we get $U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)} < b_h^{\mathbf{r}}$ and, consequently, $a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}$, being in contradiction with the hypothesis.

Now, let $b_h^{\mathbf{r}} < \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ and $a \in A$ an alternative, such that

$$U_{(\mathbf{r},j)}(a) = b_h^{(\mathbf{r},j)} - \frac{\varepsilon}{n(\mathbf{r})} \ \text{ for all } \ j = 1, \ldots, n(\mathbf{r}) \tag{12}$$

where $\varepsilon > 0$. This choice implies that $a \underset{(\mathbf{r},\mathbf{j})}{\longrightarrow} C_{\leq h}$, for all $j = 1, \ldots, n(\mathbf{r})$. Now, adding up with respect to j in the two members of equation (12), we get $U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} \left[b_h^{(\mathbf{r},j)} - \frac{\varepsilon}{n(\mathbf{r})} \right] = \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)} - \varepsilon$. If we choose ε such that

$$0 < \varepsilon \leq \min \left\{ \min \left\{ n(\mathbf{r}) \cdot \left[b_h^{(\mathbf{r},j)} - b_{h-1}^{(\mathbf{r},j)} \right], \ j = 1, \ldots, n(\mathbf{r}) \right\}, \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)} - b_h^{\mathbf{r}} \right\}$$

we obtain that $b_{h-1}^{(\mathbf{r},j)} \leq U_{(\mathbf{r},j)}(a) < b_h^{(\mathbf{r},j)}$ for all $j = 1, \ldots, n(\mathbf{r})$[2] and $U_{\mathbf{r}}(a) > b_h^{\mathbf{r}}$[3] implying that $a \xrightarrow[(\mathbf{r},j)]{} C_h$ for all $j = 1, \ldots, n(\mathbf{r})$ and $a \underset{\mathbf{r}}{\to} C_{\geq h+1}$, thus leading to a contradiction. □

B Proof of Proposition 4.1

Proof. 1. Let $a \in A$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ and $h = 2, \ldots, p$ such that not$\Big(a \xrightarrow[\mathbf{r}]{N} C_{\geq h}\Big)$. This means that there exists at least one (U, b) such that $U_{\mathbf{r}}(a) < b_{h-1}^{\mathbf{r}}$. Therefore $a \xrightarrow[\mathbf{r}]{P} C_{\leq h-1}$. Let us observe that $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ and $a \xrightarrow[\mathbf{r}]{P} C_{\leq h-1}$ do not hold simultaneously because, otherwise, a couple $(\overline{U}, \overline{b})$ should exist, such that $\overline{U}_{\mathbf{r}}(a) \geq \overline{b}_{h-1}^{\mathbf{r}}$ and $\overline{U}_{\mathbf{r}}(a) < \overline{b}_{h-1}^{\mathbf{r}}$, which is impossible.

2. Let $a \in A$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ and $k = 1, \ldots, p-1$ such that not$\Big(a \xrightarrow[\mathbf{r}]{N} C_{\leq k}\Big)$. This means that there exists at least one (U, b) such that $U_{\mathbf{r}}(a) \geq b_k^{\mathbf{r}}$. Therefore $a \xrightarrow[\mathbf{r}]{P} C_{\geq k+1}$. Let us observe that $a \xrightarrow[\mathbf{r}]{N} C_{\leq k}$ and $a \xrightarrow[\mathbf{r}]{P} C_{\geq k+1}$ do not hold simultaneously because, otherwise, a couple $(\overline{U}, \overline{b})$ should exist, such that $\overline{U}_{\mathbf{r}}(a) < \overline{b}_k^{\mathbf{r}}$ and $\overline{U}_{\mathbf{r}}(a) \geq \overline{b}_k^{\mathbf{r}}$, which is impossible.

3. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}$ for all $j = 1, \ldots, n(\mathbf{r})$ implies that $U_{(\mathbf{r},j)}(a) \geq b_{h_j-1}^{(\mathbf{r},j)}$ for all (U, b) and for all $j = 1, \ldots, n(\mathbf{r})$. Considering $h = \min_{j=1,\ldots,n(\mathbf{r})} h_j$, for the monotonicity of the thresholds we have that $U_{(\mathbf{r},j)}(a) \geq b_{h-1}^{(\mathbf{r},j)}$ for all (U, b) and for all j. As a consequence, adding up with respect to j, we get $U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} b_{h-1}^{(\mathbf{r},j)} = b_{h-1}^{\mathbf{r}}$ for all (U, b), which proves point 2.

4. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}$ for all $j = 1, \ldots, n(\mathbf{r})$ implies that $U_{(\mathbf{r},j)}(a) < b_{k_j}^{(\mathbf{r},j)}$ for all (U, b) and for all $j = 1, \ldots, n(\mathbf{r})$. Considering $k = \max_{j=1,\ldots,n(\mathbf{r})} k_j$, for the monotonicity of the thresholds we have that $U_{(\mathbf{r},j)}(a) < b_k^{(\mathbf{r},j)}$ for all (U, b) and for all j. As a consequence, adding up with respect to j, we get $U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) < \sum_{j=1}^{n(\mathbf{r})} b_k^{(\mathbf{r},j)} = b_k^{\mathbf{r}}$ for all (U, b), which implies point 3.

5. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}$, for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\bar{j}\}$ implies that for all (U, b), $U_{(\mathbf{r},j)}(a) \geq b_{h_j-1}^{(\mathbf{r},j)}$ for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\bar{j}\}$. Analogously, $a \xrightarrow[(\mathbf{r},\bar{j})]{P} C_{\geq h_{\bar{j}}}$ implies that there exists at least one $(\overline{U}, \overline{b})$ such that $\overline{U}_{(\mathbf{r},\bar{j})}(a) \geq \overline{b}_{h_{\bar{j}}-1}^{(\mathbf{r},\bar{j})}$. Considering $h = \min_{j=1,\ldots,n(\mathbf{r})} h_j$, for $(\overline{U}, \overline{b})$ and for the monotonicity of the thresholds we have that $\overline{U}_{(\mathbf{r},j)}(a) \geq \overline{b}_{h-1}^{(\mathbf{r},j)}$ for all $j = 1, \ldots, n(\mathbf{r})$. Adding up with respect to j we get $\overline{U}_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) \geq \sum_{j=1}^{n(\mathbf{r})} \overline{b}_{h-1}^{(\mathbf{r},j)} = \overline{b}_{h-1}^{\mathbf{r}}$, which proves point 4.

[2] Because $\varepsilon \leq \min_{j=1,\ldots,n(\mathbf{r})} n(\mathbf{r}) \left[b_h^{(\mathbf{r},j)} - b_{h-1}^{(\mathbf{r},j)}\right]$.

[3] Because $\varepsilon \leq \sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)} - b_h^{\mathbf{r}}$ and, consequently $\sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)} - \varepsilon > b_h^{\mathbf{r}}$

6. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}$, for all $j \in \{1,\dots,n(\mathbf{r})\} \setminus \{\bar{j}\}$ implies that for all (U,b), $U_{(\mathbf{r},j)}(a) < b_{k_j}^{(\mathbf{r},j)}$ for all $j \in \{1,\dots,n(\mathbf{r})\} \setminus \{\bar{j}\}$. Analogously, $a \xrightarrow[(\mathbf{r},\bar{j})]{P} C_{\leq k_{\bar{j}}}$ implies that there exists at least one $(\overline{U},\overline{b})$ such that $\overline{U}_{(\mathbf{r},\bar{j})}(a) < \overline{b}_{k_{\bar{j}}}^{(\mathbf{r},\bar{j})}$. Considering $k = \max_{j=1,\dots,n(\mathbf{r})} k_j$, for $(\overline{U},\overline{b})$ and for the monotonicity of the thresholds we have that $\overline{U}_{(\mathbf{r},j)}(a) < \overline{b}_k^{(\mathbf{r},j)}$ for all $j = 1,\dots,n(\mathbf{r})$. Adding up with respect to j we get $\overline{U}_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) < \sum_{j=1}^{n(\mathbf{r})} \overline{b}_k^{(\mathbf{r},j)} = \overline{b}_k^{\mathbf{r}}$, which proves point 5.

□

C Proof of Proposition 4.2

Proof. 1. Let $L_{(\mathbf{r},j)}^{\mathcal{U},P}(a) = h_j$ for all $j = 1,\dots,n(\mathbf{r})$. This means that $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}$ and not$\left(a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq l}\right)$ with $l > h_j$ for all $j = 1,\dots,n(\mathbf{r})$. By Proposition 4.1 we get $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ with $h = \min_{j=1,\dots,n(\mathbf{r})} h_j$. As a consequence we get the thesis.

2. Let $R_{(\mathbf{r},j)}^{\mathcal{U},P}(a) = k_j$ for all $j = 1,\dots,n(\mathbf{r})$. This means that $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}$ and not$\left(a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq l}\right)$ with $l > k_j$ for all $j = 1,\dots,n(\mathbf{r})$. By Proposition 4.1 we get $a \xrightarrow[\mathbf{r}]{N} C_{\geq k}$ with $k = \max_{j=1,\dots,n(\mathbf{r})} k_j$. As a consequence we get the thesis.

□