Stochastic Multiobjective Acceptability Analysis for the Choquet integral preference model and the scale construction problem

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Abstract

The Choquet integral preference model is adopted in Multiple Criteria Decision Aiding (MCDA) to deal with interactions between criteria, while the Stochastic Multiobjective Acceptability Analysis (SMAA) is an MCDA methodology considered to take into account uncertainty or imprecision on the considered data and preference parameters. In this paper, we propose to combine the Choquet integral preference model with the SMAA methodology in order to get robust recommendations taking into account all parameters compatible with the preference information provided by the Decision Maker (DM). In case the criteria are on a common scale, one has to elicit only a set of non-additive weights, technically a capacity, compatible with the DM's preference information. Instead, if the criteria are on different scales, besides the capacity, one has to elicit also a common scale compatible with the preferences given by the DM. Our approach permits to explore the whole space of capacities and common scales compatible with the DM's preference information.

Keywords: MCDA, Choquet integral, SMAA, interacting criteria, evaluation scales.

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1 Introduction

In Multiple Criteria Decision Aiding (MCDA) (see [9] for a collection of surveys on MCDA), an alternative a_k, belonging to a finite set of l alternatives $A = \{a_1, \ldots, a_l\}$, is evaluated on the basis of a family of n criteria $G = \{g_1, \ldots, g_n\}$. For example, in a car decision problem, the set A is composed of different car models while criteria in G are features of the cars taken into consideration, such as, price, maximum speed, acceleration and so on. In the description of the methodology we are proposing, we shall suppose, for the sake of simplicity, that $g_i : A \rightarrow X_i \subseteq \mathbb{R}$, which does not exclude X_i from being a number-coded ordinal scale.

To give a recommendation for the decision making problem at hand, the evaluations of the alternatives on all criteria have to be aggregated. In literature, the three main aggregation approaches are the Multi-Attribute Value Theory (MAVT) [26], the outranking methods [34] (among which the most well known are the ELECTRE [11] and PROMETHEE [5] methods) and the Dominance-based Rough Set Approach (DRSA, see [20]). In the following, we shall describe MAVT being the aggregation approach used in the paper.

MAVT takes into consideration an overall value function $U\colon \mathbb{R}^n \rightarrow \mathbb{R}$ with $U(g_1(a_k), \ldots, g_n(a_k)) = U(a_k)$, such that alternative a_k is indifferent to alternative a_h iff $U(a_k) = U(a_h)$ and a_k is preferred to a_h iff $U(a_k) > U(a_h)$ for any a_k, $a_h \in A$. The value functions used in MAVT can take different forms, but, the most common is the additive one. It is based on the preference independence of the criteria [26, 45], even if it is an unrealistic assumption or a too strong simplification, since in many cases the criteria can be interacting. For instance, let us consider the car decision problem introduced above. On one hand, maximum speed and acceleration are redundant criteria because, in general, speedy cars also have a good acceleration. Therefore, even if these two criteria can be very important, their comprehensive importance is smaller than the sum of the importance of the two criteria considered separately. On the other hand, maximum speed and price lead to a synergy effect, because a speed car having also a low price is very well appreciated. For such a reason, the comprehensive importance of these two criteria should be greater than the sum of the importance of the two criteria considered separately.

In the MAVT context, multiplicative and multilinear value functions are able to take into account interactions between criteria, but, due to the high number of parameters that have to be elicited from the DM, its use results of marginal relevance in real world applications [38]. Recently, interactions between criteria have been considered also in ELECTRE methods [10] and PROMETHEE methods [7].

Within MCDA, the interaction between criteria has frequently been dealt by using non-additive integrals, the most well known of which are the Choquet integral [6] and the Sugeno integral [39] (see [12, 17, 18] for a comprehensive survey on the use of non-additive integrals in MCDA; see also [15, 16, 19, 21] for some recently proposed extensions of non-additive integrals useful in MCDA).

The two main drawbacks of the Choquet integral preference model are the great number of parameters that have to be elicited in order to apply it and the requirement that criteria are on a common scale.

Regarding the elicitation of the preference parameters, the DM can provide direct or indirect preference information [3, 31]. The DM gives direct preference information when she provides directly all the values of the parameters present in the model. The DM supplies indirect preference information (see e.g. [22]) when she provides some preferences between alternatives or comparisons about importance and interaction of criteria from which compatible preference parameters can be inferred. With respect to the Choquet integral preference model, the inference of the preference parameters is really challenging, but several methodologies have been proposed in literature [14, 31].

Concerning the common scale problem, let us mention that the Choquet integral preference model requires that evaluations on different criteria have to be compared between them. For example, in the considered car decision problem, the DM should be able to compare the speed of a car with its acceleration estimating, for example, if the maximum speed of 200 km/h is as valuable as a price of $35,000$ €. This problem is quite well known in literature (see e.g. [32]) but, to the best of our knowledge, very few contributions tackled the problem (e.g. [3] proposes a search of a common scale through Monte Carlo simulation). In this paper, we shall deal with these two drawbacks of the Choquet integral preference model.

The elicitation of the preference parameters has been already taken into account in our previous work [1], where the SMAA-Choquet methodology has been presented. In that paper, we have applied the Stochastic Multiobjective Acceptability Analysis (SMAA) (for a survey on SMAA methods see [40]) to explore the whole space of parameters compatible with some preference information provided by the DM related to the importance and the interaction of criteria.

The contributions of this paper are threefold:

1. SMAA-Choquet has been extended by taking into account also the DM's preference information regarding the pairwise comparison of some reference alternatives,

2. SMAA-Choquet has been also enhanced by including the possibility that the evaluations on criteria may be given imprecisely, that is the evaluation of each alternative on the considered

criteria is not given punctually but as interval of possible evaluations,

3. SMAA-Choquet includes a procedure aiming to obtain a common scale for all considered criteria permitting, therefore, to apply the Choquet integral preference model.

The paper is organized as follows. In Section 2, we introduce the Choquet integral preference model together with a didactic example. In Section 3, we briefly describe the SMAA methods. Our simulation based approach, proposed in the context of the Choquet integral preference model, is introduced in Section 4 and illustrated by two examples in Section 5. Some conclusions and future directions of research are presented in Section 6.

2 The Choquet integral preference model

Very often the aggregation of the evaluations of an alternative on the considered criteria is done by means of the simplest additive value function, i.e. the *weighted sum*. It is obtained considering a vector of non-negative weights $\mathbf{w} = [w_1, ..., w_n]$ (one for each criterion in G), that permits to assign a value $U(a_k) = w_1 g_1(a_k) + \ldots + w_n g_n(a_k)$ to the alternative $a_k \in A$. Notice that, in the rest of the paper, we shall use the terms criterion g_i and criterion i interchangeably.

The weighted sum has several limitations to represent preferences (see e.g. [12, 26]) as illustrated by the following didactic example inspired by [17].

Example The dean of a technical school wants to evaluate students s_1, s_2 and s_3 whose marks on Mathematics and Physics are shown in Table 1.

Table 1: Students' evaluations on Mathematics and Physics given on a $[0, 30]$ scale

	Math	Phy
s_1	26	30
s_2	28	28
s_3	30	26

Since students good in Mathematics are in general good also in Physics, if there is a good mark in one of the two subjects, one can expect a good mark also in the other subject. Consequently, a student good in Mathematics and in Physics is of course appreciated, but the dean does not want to overvalue students having good marks in both subjects. Thus, for the dean, students s_1 and s_3 are preferred to student s_2.

In this case, we can say that there is a negative interaction (redundancy) between Mathematics and Physics. To represent the dean's preferences by means of the weighted sum model, the following inequalities should be satisfied:

$$\begin{aligned} w_{\mathrm{Math}} \cdot 26 + w_{\mathrm{Phy}} \cdot 30 &> w_{\mathrm{Math}} \cdot 28 + w_{\mathrm{Phy}} \cdot 28, \\ w_{\mathrm{Math}} \cdot 30 + w_{\mathrm{Phy}} \cdot 26 &> w_{\mathrm{Math}} \cdot 28 + w_{\mathrm{Phy}} \cdot 28, \end{aligned}$$

where w_{Math} and w_{Phy} are the weights of Mathematics and Physics, respectively. It is easily verified that the above inequalities are contradictory since:

$$w_{\mathrm{Math}} \cdot (-2) + w_{\mathrm{Phy}} \cdot 2 > 0 > w_{\mathrm{Math}} \cdot (-2) + w_{\mathrm{Phy}} \cdot 2$$

Thus, we have to conclude that, due to the redundancy between Mathematics and Physics, the weighted sum is not able to represent the dean's preferences. $\square$

In order to represent preferences in case of interaction between criteria, one has to use some preference model more general than the weighted sum. This is the case of the non-additive integrals among which the most well-known is the Choquet integral [6]. It proposes an extension of the weighted sum model to the case of interacting criteria and it is based on the concept of capacity (fuzzy measure) that assigns a weight to each subset of criteria. More precisely, denoting by 2^G the power set of G (i.e. the set of all subsets of G), the function $\mu : 2^G \rightarrow [0,1]$ is called a capacity (fuzzy measure) on 2^G if the following properties are satisfied:

1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions),

2a) $\forall\, S \subseteq T \subseteq G,\ \ \mu(S) \leq \mu(T)$ (monotonicity condition).

Intuitively, for all $T \subseteq G$, $\mu(T)$ can be interpreted as the comprehensive importance of the criteria from T considered as a whole.

Example (Continuation). To represent the importance of Mathematics and Physics taken singularly and considered together, one can set $\mu_1(\{Math\}) = \mu_1(\{Phy\}) = 0.6$ and $\mu_1(\{Math, Phy\}) = 1$. The difference $\mu_1(\{Math, Phy\}) - \mu_1(\{Math\}) - \mu_1(\{Phy\}) = -0.2$ represents the negative interaction between Mathematics and Physics since it is the difference between the importance of Mathematics and Physics considered as a whole ($\mu_1(\{Math, Phy\})$), and the sum of their importance when they are considered singularly ($\mu_1(\{Math\}) + \mu_1(\{Phy\})$). $\square$

If there is no interaction between the considered criteria, we have $\mu(S \cup T) = \mu(S) + \mu(T)$, for any $S, T \subseteq G$ such that $S \cap T = \emptyset$ and the capacity is called *additive*. If a capacity is additive

then $\mu(T) = \sum_{i \in T} \mu(\{i\})$ and, consequently, the values $\mu(\{1\}), \mu(\{2\}) \ldots, \mu(\{n\})$ (corresponding to the weights w_i of the weighted sum model), are sufficient to rebuild the whole capacity μ.
Whenever the capacity is non-additive, in general, one has to assess $2^{|G|} - 2$ values $\mu(T), \emptyset \subset T \subset G$, since the values $\mu(\emptyset) = 0$ and $\mu(G) = 1$ are already known.

If the criteria from G are interacting and their importance is represented by a capacity μ, the weighted sum can be extended through the Choquet integral [6] that assigns the following value to each $a_k \in A$:

$$C_\mu(a_k) = \sum_{i=1}^{n} \left[g_{(i)}(a_k) - g_{(i-1)}(a_k)\right] \mu\left(N_i\right),$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that $g_{(1)}(a_k) \leq \ldots \leq g_{(n)}\left(a_k\right)$, $N_i = \{(i), \ldots, (n)\}$ and $g_{(0)} = 0$.

A meaningful and useful reformulation of the capacity μ and of the Choquet integral can be obtained by means of the Möbius representation of the capacity μ which is a function $m : 2^G \to \mathbb{R}$ [35] defined as follows:

$$\mu(S) = \sum_{T \subseteq S} m(T).$$

Note that if S is a singleton, i.e. $S = \{i\}$ with $i = 1, 2, \ldots, n$, then $\mu(\{i\}) = m(\{i\})$ while, if S is a couple (non-ordered pair) of criteria, i.e. $S = \{i, j\}$, then $\mu(\{i, j\}) = m(\{i\}) + m(\{j\}) + m(\{i, j\})$.
The Möbius representation $m(S)$ can be obtained from $\mu(S)$ as follows:

$$m(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \mu(T).$$

In terms of Möbius representation, properties **1a)** and **2a)** are, respectively, restated as:

1b) $m(\emptyset) = 0, \ \sum_{T \subseteq G} m(T) = 1,$

2b) $\forall\, i \in G$ and $\ \forall R \subseteq G \setminus \{i\}, \ m(\{i\}) + \sum_{T \subseteq R} m(T \cup \{i\}) \geq 0.$

The Choquet integral may be reformulated in terms of Möbius representation as follows:

$$C_\mu(a_k) = \sum_{T \subseteq G} m(T) \min_{i \in T} g_i\left(a_k\right). \tag{1}$$

Example (Continuation). The value assigned to student s_1 by the Choquet integral in terms of the capacity μ_1 is the following:

$$C_{\mu_1}(s_1) = g_{Math}(s_1) \cdot \mu_1(\{Math, Phy\}) + (g_{Phy}(s_1) - g_{Math}(s_1)) \cdot \mu_1(\{Phy\}) = 28.4.$$

This value can be explained as follows. The mark $g_{Math}(s_1) = 26$ is attained on both subjects and, therefore, it is multiplied by $\mu_1(\{Math, Phy\})$, i.e. the weight assigned to Mathematics and Physics considered as a whole. The mark $g_{Phy}(s_1) = 28$ is attained on Physics only and, consequently, the difference $g_{Phy}(s_1) - g_{Math}(s_1)$ is multiplied by $\mu_1(\{Phy\})$, i.e. the weight assigned to Physics considered singularly. Analogously, we get $C_{\mu_1}(s_2) = 28$ and $C_{\mu_1}(s_3) = 28.4$, so that $C_{\mu_1}(s_1) > C_{\mu_1}(s_2)$ and $C_{\mu_1}(s_3) > C_{\mu_1}(s_2)$. Therefore, we can conclude that the Choquet integral is able to represent the dean's preferences.

Observe also that the Möbius representation m_1 of the capacity μ_1 gives $m_1(\{Math\}) = m_1(\{Phy\}) = 0.6$ and $m_1(\{Math, Phy\}) = -0.2$ and, consequently, the Choquet integral related to student s_1 can be reformulated as follows in terms of the Möbius representation m_1:

$$C_{\mu_1}(s_1) = g_{Math}(s_1) \cdot m_1(\{Math\}) + g_{Phy}(s_1) \cdot m_1(\{Phy\}) + \min(g_{Math}(s_1), g_{Phy}(s_1)) \cdot m_1(\{Math, Phy\}) = 28.4$$

Considering its formulation in terms of Möbius representation, the Choquet integral can be explained as follows. The marks in Mathematics and Physics are multiplied by $m_1(\{Math\})$ and $m_1(\{Phy\})$ representing, in some form, the weights related to their additive components. However, the value so obtained has to be corrected by adding $\min(g_{Math}(s_1), g_{Phy}(s_1)) \cdot m_1(\{Math, Phy\})$ representing the negative interaction between Mathematics and Physics. The Choquet integral related to students s_2 and s_3 can be analogously reformulated in terms of the Möbius representation m_1. $\square$

With the aim of reducing the number of parameters to be elicited, in [13] the concept of k-additive capacity has been introduced. A capacity is called k-additive if $m(T) = 0$ for $T \subseteq G$ such that $|T| > k$.

Within an MCDA context, it is easier and more straightforward to consider 2-additive capacities since, in such case, the DMs have to express a preference information on positive and negative interactions between two criteria, neglecting more complex interactions among three, four and generally $k \leq n$ criteria. Moreover, by considering 2-additive measures the computational issue of determining the parameters is weakened, since only $n + \binom{n}{2}$ coefficients have to be assessed; specifically, in terms of Möbius representation, a value $m(\{i\})$ for every criterion i and a value $m(\{i, j\})$ for every couple of criteria $\{i, j\}$. For all these reasons, in the following we shall consider 2-additive capacities only. However, the methodology we are presenting can be applied to any capacity.

The value that a 2-additive capacity μ assigns to a set $S \subseteq G$ can be expressed in terms of the Möbius representation as follows:

$$\mu(S) = \sum_{i \in S} m(\{i\}) + \sum_{\{i,j\} \subseteq S} m(\{i,j\}), \ \forall S \subseteq G.$$

With regard to 2-additive capacities, properties **1b)** and **2b)** have, respectively, the following expressions:

1c) $m(\emptyset) = 0, \ \sum_{i \in G} m(\{i\}) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) = 1,$

2c) $\begin{cases} m(\{i\}) \geq 0, \ \forall i \in G, \\ m(\{i\}) + \sum_{j \in T} m(\{i,j\}) \geq 0, \ \forall i \in G \text{ and } \forall\, T \subseteq G \setminus \{i\}, T \neq \emptyset. \end{cases}$

In this case, the Choquet integral assigns to $a_k \in A$ the following value:

$$C_\mu(a_k) \quad = \quad \sum_{i \in G} m(\{i\})\, g_i(a_k) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) \min\{g_i(a_k), g_j(a_k)\}. \tag{2}$$

Since, in this context, the importance of a criterion does not depend only on its importance as a single but also on its contribution to each coalition of criteria to which it participates, we recall the definitions of the importance of a criterion and of the interaction index for a couple of criteria. Taking into account the Möbius representation of a 2-additive capacity μ, the importance of criterion $i \in G$, expressed by the Shapley value [36], can be written as follows:

$$\varphi(\{i\}) = m(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{m(\{i,j\})}{2}.$$

The interaction index, expressing the sign and the magnitude of the interaction in a couple of criteria $\{i,j\} \subseteq G$ in case of a 2-additive Möbius representation of a capacity μ, is given by:

$$\varphi(\{i,j\}) = m(\{i,j\}).$$

Example (Continuation). Capacity μ_1 is trivially 2-additive and we have

$$\varphi(\{Math\}) = m_1(\{Math\}) + \frac{m_1(\{Math, Phy\})}{2} = 0.5.$$

Observing that $\varphi(\{Phy\}) = 0.5$ and, consequently, $\varphi(\{Math\}) = \varphi(\{Phy\})$, we can conclude that the marks in Mathematics and Physics have the same importance. Moreover, the value $\varphi(\{Math, Phy\}) = -0.2$ confirms that the two considered criteria are negatively interacting. $\square$

3 SMAA

Stochastic Multiobjective Acceptability Analysis (SMAA) [27, 29] is a family of MCDA methods aiming to get recommendations on the problem at hand taking into account uncertainty or imprecision on the considered data and preference parameters. Several SMAA methods have been developed to deal with different MCDA problems: SMAA-2 has been presented in [29] for ranking problems, SMAA-O [28] has been introduced for multicriteria problems with ordinal criteria and SMAA-TRI [41] for sorting problems. Other two recent contributions related to SMAA and ROR have been presented in [24] and [25]. In the following, we shall describe SMAA-2 since, in this paper, we have considered ranking problems only.

In SMAA-2, the most commonly used value function is the linear one:

$$u(a_k, w) = \sum_{i=1}^{n} w_i g_i(a_k).$$

In order to take into account imprecision or uncertainty, SMAA-2 considers two probability distributions $f_W(w)$ and $f_\chi(\xi)$ on W and χ, respectively, where $W = \{(w_1, \ldots, w_n) \in \mathbb{R}^n \colon w_i \geq 0 \text{ and } \sum_{i=1}^{n} w_i = 1\}$ and χ is the evaluation space.

First of all, SMAA-2 introduces a ranking function relative to the alternative a_k:

$$rank(k, \xi, w) = 1 + \sum_{h \neq k} \rho\left(u(\xi_h, w) > u(\xi_k, w)\right),$$

where $\rho(false) = 0$ and $\rho(true) = 1$.
Then, for each alternative a_k, for each evaluation of alternatives $\xi \in \chi$ and for each rank $r = 1, \ldots, l$, SMAA-2 computes the set of weights of criteria for which alternative a_k assumes rank r:

$$W_k^r(\xi) = \{w \in W : rank(k, \xi, w) = r\}.$$

SMAA-2 is based on the computation of the following indices:

- *The rank acceptability index*, which measures the variety of different parameters compatible with the DM's preference information giving to the alternative a_k the rank r:

  $$b_k^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W_k^r(\xi)} f_W(w)\, dw\, d\xi;$$

  b_k^r gives the probability that alternative a_k has rank k and it is within the range $[0, 1]$.

- *The central weight vector* , which describes the preferences of a typical DM giving to a_k the best position and it is defined as follows:

$$w_k^c = \frac{1}{b_k^1}\int_{\xi\in\chi} f_\chi(\xi)\int_{w\in W^1(\xi)} f_W(w)w\; dw\; d\xi;$$

- *The confidence factor*, which is defined as the frequency of an alternative to be the preferred one with the preferences expressed by its central weight vector and it is given by:

$$p_k^c = \int_{\substack{\xi\in\chi:u(\xi_k,w_k^c)\geq u(\xi_h,w_k^c)\\ \forall h=1,\ldots,l}} f_\chi(\xi)\; d\xi.$$

In the following, we shall consider also the frequency that an alternative a_h is weakly preferred to an alternative a_k in the space of the preference parameters (weight vectors in case of SMAA-2):

$$p_{hk} = \int_{w\in W} f_W(w)\int_{\xi\in\chi:u(\xi_h,w)\geq u(\xi_k,w)} f_\chi(\xi)d\xi\; dw.$$

Let us notice that the previous index p_{hk} is also known as pairwise winning index and it has been introduced in [30, 42].

From a computational point of view, the multidimensional integrals defining the considered indices are estimated by using the Monte Carlo method.

4 An extension of the SMAA method to the Choquet integral preference model

In this section, we shall present the SMAA-Choquet method putting together the Choquet integral preference model and the SMAA methodology.

As observed in Section 2, the use of the Choquet integral in terms of Möbius representation with a 2-additive capacity requires the evaluation of $n+\binom{n}{2}$ parameters and in order to assess these parameters, the DM is asked to provide some preference information in a direct or an indirect way. Generally, the indirect preference information requires less cognitive effort from the DM, and for this reason it is widely used in MCDA (see for example [4, 20, 22, 23]). In the following, we shall suppose that the DM is able to provide some indirect preference information and we shall use the 2-additive Choquet integral preference model expressed in terms of the Möbius representation.

Using the Choquet integral preference model, the DM can provide the following preference information:

- Comparisons related to importance and interaction of criteria:
  - criterion i is at least as important as criterion j (and we shall write $i \succsim j$): $\varphi(\{i\}) \geq \varphi(\{j\})$;
  - criterion i is more important than criterion j ($i \succ j$): $\varphi(\{i\}) > \varphi(\{j\})$;
  - criteria i and j have the same importance ($i \sim j$): $\varphi(\{i\}) = \varphi(\{j\})$;
  - criteria i and j are synergic: $\varphi(\{i,j\}) > 0$;
  - criteria i and j are redundant: $\varphi(\{i,j\}) < 0$.

- Comparisons between couples or quadruples of alternatives:
  - alternative a_k is at least as good as alternative a_h ($a_k \succsim a_h$): $C_\mu(a_k) \geq C_\mu(a_h)$;
  - alternative a_k is preferred to alternative a_h ($a_k \succ a_h$): $C_\mu(a_k) > C_\mu(a_h)$;
  - alternative a_k and a_h are indifferent ($a_k \sim a_h$): $C_\mu(a_k) = C_\mu(a_h)$;
  - alternative a_k is preferred to alternative a_h more than alternative a_s is preferred to alternative a_t ($(a_k, a_h) \succ^* (a_s, a_t)$): $C_\mu(a_k) - C_\mu(a_h) > C_\mu(a_s) - C_\mu(a_t)$;
  - the difference of preference between a_k and a_h is the same of the difference of preference between a_s and a_t ($(a_k, a_h) \sim^* (a_s, a_t)$): $C_\mu(a_k) - C_\mu(a_h) = C_\mu(a_s) - C_\mu(a_t)$.

Hereafter, we distinguish three sets of constraints:

- Monotonicity and boundary constraints,

$$\left.\begin{array}{l}
m\left(\{\emptyset\}\right) = 0, \ \sum\limits_{i \in G} m\left(\{i\}\right) + \sum\limits_{\{i,j\} \subseteq G} m\left(\{i,j\}\right) = 1, \\
m\left(\{i\}\right) \geq 0, \ \forall i \in G, \\
m\left(\{i\}\right) + \sum\limits_{j \in T} m\left(\{i,j\}\right) \geq 0, \forall i \in G \text{ and } \forall\, T \subseteq G \setminus \{i\}, T \neq \emptyset,
\end{array}\right\} (E^{MB})$$

- Constraints related to importance and interaction of criteria,

$$\left.\begin{array}{ll}
\varphi(\{i\}) \geq \varphi(\{j\}), & \text{if } i \succsim j, \\
\varphi(\{i\}) \geq \varphi(\{j\}) + \varepsilon, & \text{if } i \succ j, \\
\varphi(\{i\}) = \varphi(\{j\}), & \text{if } i \sim j, \\
\varphi(\{i,j\}) \geq \varepsilon, & \text{if criteria } i \text{ and } j \text{ are synergic with } i, j \in G, \\
\varphi(\{i,j\}) \leq -\varepsilon, & \text{if criteria } i \text{ and } j \text{ are redundant with } i, j \in G,
\end{array}\right\} (E^{C})$$

- Constraints related to comparisons between alternatives,

$$\left.\begin{array}{l} C_\mu(a_k) \geq C_\mu(a_h), \quad \text{if } a_k \succsim a_h, \\ C_\mu(a_k) \geq C_\mu(a_h) + \varepsilon, \quad \text{if } a_k \succ a_h, \\ C_\mu(a_k) = C_\mu(a_h) \quad \text{if } a_k \sim a_h, \\ C_\mu(a_k) - C_\mu(a_h) \geq C_\mu(a_s) - C_\mu(a_t) + \varepsilon, \quad \text{if } (a_k, a_h) \succ^* (a_s, a_t), \\ C_\mu(a_k) - C_\mu(a_h) = C_\mu(a_s) - C_\mu(a_t), \quad \text{if } (a_k, a_h) \sim^* (a_s, a_t), \end{array}\right\} (E^A)$$

where the strict inequalities used to translate the preferences have been transformed into weak inequalities in E^C and E^A by adding an auxiliary variable ε taking positive values.

We shall call *compatible model*, a capacity whose Möbius representation satisfies the set of constraints $E^{DM} = E^{MB} \cup E^C \cup E^A$ with a positive value of ε. Observe that E^C or E^A could be eventually empty if the DM does not provide any information on importance and interaction of criteria, or comparison of alternatives, respectively.

In order to check if there exists at least one compatible model, one has to solve the following linear programming problem:

$$\begin{array}{l} \max \varepsilon = \varepsilon^* \quad s.t. \\ E^{DM}. \end{array} \tag{3}$$

If E^{DM} is feasible and $\varepsilon^* > 0$, then there exists at least one model compatible with the preference information provided by the DM. If E^{DM} is infeasible or $\varepsilon^* \leq 0$, then one can check which is the minimum set of constraints determining the infeasibility using one of the techniques described in [33].

In this section, we shall describe how to obtain robust recommendations on the problem at hand by putting together the Choquet integral preference model and the SMAA methodology that is, by estimating the indices typical of SMAA, but considering as preference model the Choquet integral instead of an additive value function. We shall consider the following different cases:

case 1) the evaluations on the criteria are on a common scale and they are expressed in a precise way, that is $g_i(a_k) \in \mathbb{R}$ for all i and for all k,

case 2) the evaluations on criteria are on a common scale but they can be given in an imprecise way, that is $g_i(a_k) \in [\alpha_i^k, \beta_i^k]$ with $\alpha_i^k \leq \beta_i^k$, for some i and for some k,

case 3) the evaluations on the criteria are on different scales (for the sake of simplicity in this case we have supposed that evaluations of alternatives on the considered criteria are given in a precise way).

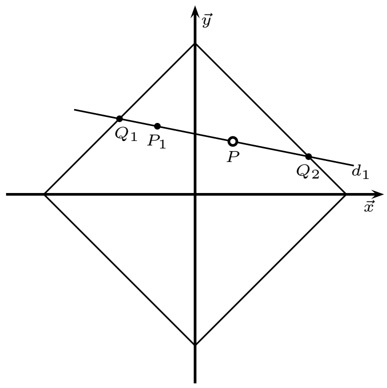
In **case 1)**, since the evaluations on the criteria under consideration are on a common scale and they are given in a precise way, the application of the Choquet integral depends only on a capacity compatible with the preferences expressed by the DM. Because the set of inequalities in E^{DM} defines a convex set of parameters, one can use the Hit-and-Run (HAR) method in order to sample some compatible models. The HAR sampling has been firstly introduced in [37] and recently applied in multicriteria decision analysis in [44]. It starts from the choice of one point (the Möbius representation of one capacity in the problem at hand) inside the polytope E^{DM}. Since the starting point in the HAR sampling could be whichever point inside the polytope, we can begin from the point obtained by solving the linear optimization problem defined in (3). At each iteration, a random direction is sampled from the unit hypersphere that, passing through the starting point, generates a line. Finally, one point inside the segment whose extremes are the intersection of the line with the boundaries of the polytope is sampled.

In order to illustrate the procedure, we shall provide the first two iterations of the Hit-and-Run algorithm in a didactic example. Let us suppose we have to sample some points (x, y) inside the region delimited by the constraints $y \leq x + 2$, $y \geq x - 2$, $y \leq -x + 2$ and $y \geq -x - 2$ (see Figure 1).

Chosen the starting point P and a vector belonging to the unit sphere of center $(0, 0)$ and radius equal to one that defines the direction d, we consider the line d_1 in Figure 2 having the direction of d and passing through the starting point P. d_1 "hits" the boundaries $y = x + 2$ and $y = -x + 2$ in the points Q_1 and Q_2, respectively. We then "run" along the segment Q_1Q_2, sampling in a uniform way the point P_1 at the first iteration. In the second iteration, the procedure continues considering P_1 as the starting point. Taking randomly a direction d, the line d_2 passing through point P_1 and having the same direction of d intersects the lines $y = x + 2$ and $y = x - 2$ in the points Q_3 and Q_4. Point P_2 is then chosen in a uniform way inside the segment connecting Q_3 and Q_4 (see Figure 3). The algorithm continues until the stopping rule (in our case the maximum number of iterations) is satisfied.

Let us observe that at each iteration of the HAR algorithm a compatible model is sampled and therefore stored. Consequently, by applying the Choquet integral preference model with each of the stored models, one can get one different ranking and, in the end, can estimate the indices typical of the SMAA methodology.

Figure 1: Hit-and-Run example



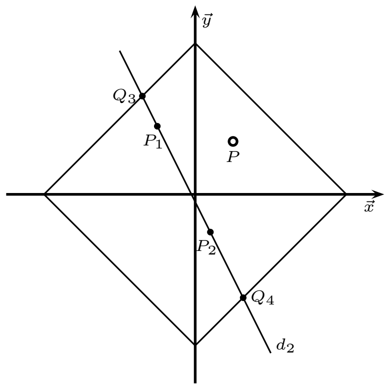


Figure 2: First iteration Figure 3: Second iteration

In **case 2)**, the application of the Choquet integral preference model does not depend on the sampled capacity only, but also on the evaluations of the alternatives at hand, because while constraints in E^C and in E^{MB} are not dependent on the alternatives' evaluations, constraints in E^A are dependent on these evaluations. Consequently, we have to distinguish between the case in which the DM does not provide any preference in terms of comparison between alternatives ($E^A = \emptyset$) from the case in which the DM expresses such type of preference ($E^A \neq \emptyset$).
If $E^A = \emptyset$, the set of constraints E^{DM} defines a convex set and therefore one can sample compatible models by applying the HAR method as described in the first case. The only difference with respect to **case 1)** is that, in order to apply the Choquet integral preference model, one has to sample an evaluation matrix M (whose element M_{ki} is taken in a random way inside the interval $[\alpha_i^k, \beta_i^k]$) for each stored capacity. After applying the Choquet integral preference model with the considered matrices and sampled capacities, one can compute the corresponding rankings and therefore estimating the SMAA indices.

Differently from the previous case, at each sampled evaluation matrix M corresponds a different set of constraints E^{DM}. Consequently, one can not apply the HAR method to sample the compatible capacities from E^A. Besides, sampled an evaluation matrix M, it is also possible that the corresponding set of constraints E^{DM} is infeasible. For this reason, after that an evaluation matrix has been sampled, one has to check if the set E^{DM} is feasible and, in this hypothesis, sampling a capacity compatible with the DM's preferences. Also in this case after storing the different rankings obtained by applying the Choquet integral with the sampled evaluations matrices and the corresponding sampled capacities, one can compute the SMAA indices.

A typical example of **case 3)** can be the evaluation of a sport car, where criteria such as maximum speed, acceleration, price, comfort can be considered and each of them has a different scale. In this case, one can not apply directly the Choquet integral to aggregate the preferences of the DM since, as remarked in the introduction, a requisite of the method is that all considered criteria are on a common scale.

In order to cope with this drawback, we propose to construct a common scale with a procedure composed of the following steps for each criterion g_i:

- sampling uniformly from the interval $[0, 1]$, l' different real numbers $x_1, \ldots, x_{l'}$ supposing that the different evaluations on g_i are l', with $l' \leq l$,
- ordering the l' numbers in an increasing way, $x_{i(1)} < \ldots < x_{i(l')}$,
- assigning $x_{i(h)}$ to the alternatives having the h-th evaluation, in an increasing order with respect to the DM's preferences on g_i.

Supposing to deal with the aforementioned car decision problem, and looking at the evaluations of the considered cars on criterion acceleration shown in Table 2, we proceed as follows:

- Because the evaluations of the 10 alternatives on criterion acceleration are all different, we sample 10 different real numbers from the interval $[0, 1]$. For example, $x_1 = 0.81$, $x_2 = 0.90$, $x_3 = 0.12$, $x_4 = 0.91$, $x_5 = 0.63$, $x_6 = 0.09$, $x_7 = 0.27$, $x_8 = 0.54$, $x_9 = 0.95$, $x_{10} = 0.96$.
- We order the 10 numbers in an increasing way: $x_{(1)} = 0.09 < x_{(2)} = 0.12 < x_{(3)} = 0.27 < x_{(4)} = 0.54 < x_{(5)} = 0.63 < x_{(6)} = 0.81 < x_{(7)} = 0.90 < x_{(8)} = 0.91 < x_{(9)} = 0.95 < x_{(10)} = 0.96$.
- Since, in this example, acceleration has a decreasing direction of preference (the lower the evaluation on the criterion, the better the alternative is) we assign value $x_{(1)} = 0.09$ to SEAT Ibiza ST 1.2, value $x_{(2)} = 0.12$ to SKODA Fabia 1.2 and so on (see the third column of Table 2).

The values $x_{i(r)}, i = 1, \ldots, n$ and $r = 1, \ldots, l'$, become the evaluations of the considered alternatives on the different criteria. In this way, evaluations on all criteria are expressed on the same common scale and therefore, having a capacity compatible with the DM's preferences, one can compute the Choquet integral of all alternatives.

At this point, since the sampling of a compatible model will depend on the chosen common scale only, if $E^A \neq \emptyset$ (the DM provides some preference on the considered alternatives), one can

Table 2: Car evaluation with respect to the criterion acceleration (expressed in seconds necessary to reach 100 Km/h starting from 0 Km/h) and the corresponding scale

Cars	Acceleration 0/100 km/h	Scale value
PEUGEOT 208 1.6 8V	10.9	0.96
Citroen C3	13.5	0.54
FIAT 500 0.9	11	0.95
SKODA Fabia 1.2	14.2	0.12
LANCIA Ypsilon 5p	11.4	0.90
RENAULT Clio 1.5 dCi 90	11.3	0.91
SEAT Ibiza ST 1.2	14.6	0.09
ALFA ROMEO MiTo 1.3	12.9	0.63
TOYOTA Yaris 1.5	11.8	0.81
VOLKSWAGEN Polo 1.2	13.9	0.27

proceed as already described in **case 2)**, but replacing the sampling of an evaluation matrix with the construction of a common scale. The only difference with **case 2)** is that the DM could be interested in discovering which is the most discriminant common scale. With this aim, one can proceed as follows:

- Sampling a certain number of possible common scales $S_1, \ldots, S_{iter}$, considering the corresponding feasible sets of constraints $E_1^{DM}, \ldots, E_{iter}^{DM}$ and denoted by $\varepsilon_1, \ldots, \varepsilon_{iter}$, the solutions of the linear programming problems

$$\left.\begin{array}{ll}\max \varepsilon & s.t.\\ E_1^{DM} & \end{array}\right\}, \ldots\ldots\ldots\ldots, \quad \left.\begin{array}{ll}\max \varepsilon & s.t.\\ E_{iter}^{DM} & \end{array}\right\}; \tag{4}$$

  the most discriminant scale is the scale S_k such that $\varepsilon_k = \max\{\varepsilon_1, \ldots, \varepsilon_{iter}\}$.

After obtaining the most discriminant common scale, the decision aiding process can continue in one of the following ways:

- applying the Choquet integral preference model after asking the capacities directly to the DM,

- eliciting one (arbitrary) capacity compatible with the DM's preference information [31],

- considering the whole set of capacities compatible with the DM's preference information using NAROR [4],

- applying the simulation techniques proposed in **case 1)** since the common scale's values become the evaluations of the alternatives on the considered criteria.

5 Some examples

The whole methodology presented in the previous section will be illustrated by two didactic examples. In the following, we shall consider uniform probability distributions f_W and f_χ, respectively, on W and χ.

5.1 Considering imprecision in the evaluations on considered criteria

Let us consider a set of 18 alternatives evaluated on the basis of 4 criteria, $G = \{g_1, g_2, g_3, g_4\}$, as shown in Table 3. We suppose that the evaluations of considered alternatives on each criterion are integer numbers within an interval (for example, the evaluation of a_1 on criterion g_1 can be 14, 15 or 16), but this is not a specific requirement for our model i.e., in general, we can sample values from the whole interval. We can consider this as a specific probability distribution $f_\chi(\xi)$ concentrating uniformly the mass only on the integers in the interval of evaluations on considered criteria.

Table 3: Imprecise evaluations of alternatives on considered criteria

	Alternatives								
Criteria	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
g_1	[14,16]	[6,8]	[17,19]	[8,10]	[11,13]	[7,9]	[13,15]	[7,9]	3
g_2	[11,13]	[7,9]	[7,9]	[15,17]	5	3	[18,20]	[12,14]	[16,18]
g_3	[9,11]	[13,15]	4	[3,5]	[13,15]	[6,9]	5	[14,15]	2
g_4	[6,9]	[15,17]	[11,13]	[15,17]	[13,15]	[18,20]	[9,11]	6	[13,15]
Criteria	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
g_1	4	[15,17]	[7,9]	[16,18]	[7,9]	[18,20]	[11,13]	[13,15]	[8,10]
g_2	[18,20]	7	[10,12]	[11,13]	[6,8]	[6,9]	4	[10,12]	[12,14]
g_3	[7,9]	[13,15]	5	[5,7]	[6,9]	[3,5]	[14,16]	[11,13]	[11,13]
g_4	[8,10]	[9,11]	[18,20]	8	[18,20]	[11,13]	[12,15]	[8,10]	[5,7]

We shall take into account the following preference information in terms of importance and interaction of criteria and comparisons between alternatives:

- $\varphi(\{g_1\} > \varphi(\{g_2\}), \varphi(\{g_3\} > \varphi(\{g_4\}),$
- $\varphi(\{g_1, g_2\}) > 0, \varphi(\{g_2, g_3\}) > 0, \varphi(\{g_2, g_4\}) < 0,$

- $a_{16} \succ a_2$, $a_3 \succ a_{14}$ and $a_{13} \succ a_8$.

According to [43], we perform the Hit-and-Run procedure for $10,000$ iterations.
For each iteration, we sample an evaluation matrix and we check if it is compatible with the preference information provided by the DM. In this case, we compute the Choquet integral for each alternative obtaining a complete ranking.
At the end of all iterations, we compute the rank acceptability index b_k^r for each $k, r = 1, \ldots, l$ and the Möbius representation of the central capacity for each alternative a_k that can get the first rank at least once, as shown, respectively, in Tables 4 and 5. In particular, in Table 4 we observe that alternatives a_1, a_3, a_7, a_{11}, a_{13}, a_{15}, a_{16} and a_{17} can be ranked first. a_{17} has reached the first position more than all other alternatives ($b_{17}^1 = 25.39$) and a_9 is instead the alternative that is almost always in the last position in the obtained rankings ($b_9^{18} = 99.52$).

Table 4: Rank acceptability indices taking into account imprecise evaluations of alternatives on considered criteria, preference information in terms of importance and interaction of criteria and comparisons between alternatives

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}	b_k^{14}	b_k^{15}	b_k^{16}	b_k^{17}	b_k^{18}
a_1	19.69	23.35	22.59	15.98	8.81	5.08	2.51	1.08	0.63	0.21	0.06	0	0.01	0	0	0	0	0
a_2	0	0.02	0.03	0.06	0.34	1.52	4.23	7.25	11.2	15.38	17.62	18.27	12.54	7.19	3.76	0.55	0.04	0
a_3	0.59	1.2	2.63	3.16	6.11	9.72	13.18	13.16	13.09	12.37	11.4	9.73	2.84	0.73	0.06	0.03	0	0
a_4	0	0.01	0.01	0.07	0.11	0.43	1.27	2.38	3.16	4.9	6.28	8.5	14.2	21.1	24.08	11.69	1.81	0
a_5	0.6	1.64	3.62	6.82	10.98	12.48	12.82	12.96	12.69	9.64	7.66	4.64	2.31	0.9	0.23	0.01	0	0
a_6	0	0	0	0	0.01	0	0.03	0.04	0.03	0.18	0.32	1.05	2.2	5.32	10.87	40.02	39.47	0.46
a_7	24.68	15.79	15.99	18	10.99	6.34	3.14	2.35	1.49	0.72	0.35	0.07	0.06	0.03	0	0	0	0
a_8	0	0	0.08	0.43	1.77	7.96	11.39	13.81	12.24	12.92	12.12	9.81	7.74	5.29	3.45	0.97	0.02	0
a_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0.47	99.52
a_{10}	0	0	0	0	0	0	0.01	0	0.02	0.02	0.12	0.2	0.66	2.32	6.87	32.5	57.26	0.02
a_{11}	23.82	21.83	17.88	15.05	9.89	5.76	3.6	1.69	0.37	0.05	0.06	0	0	0	0	0	0	0
a_{12}	0	0	0.01	0	0.01	0.11	0.25	0.64	1.85	3.35	5.44	9.08	16.84	26.49	27.61	7.86	0.46	0
a_{13}	2.27	6.37	10.45	15.96	24.34	17.15	11.08	6.36	3.79	1.47	0.57	0.15	0.03	0.01	0	0	0	0
a_{14}	0	0	0	0	0	0.05	0.34	1.31	3.06	5.57	8.87	13.99	24.1	20.09	17.52	4.76	0.34	0
a_{15}	2.49	4	4.99	7.5	9.91	13.23	12.67	10.32	8.65	6.92	6.43	5.88	3.27	2.18	1.05	0.4	0.11	0
a_{16}	0.47	1.16	1.65	2.9	5.14	9.15	12.77	16.41	16.85	14.64	10.26	5.19	2.36	1	0.05	0	0	0
a_{17}	25.39	24.59	19.83	12.91	7.95	4.45	2.96	1.19	0.58	0.13	0.02	0	0	0	0	0	0	0
a_{18}	0	0.04	0.24	1.16	3.64	6.57	7.75	9.05	10.3	11.53	12.42	13.44	10.84	7.35	4.45	1.2	0.02	0

Looking at the second best alternative, one can be in doubt among a_{11}, a_7 and a_1. In fact, on one side a_7 has a first rank acceptability index greater than the other two alternatives ($b_7^1 = 24.68\%$). On the other side, looking at the pairwise winning indices shown in Table 6, one can observe that

Table 5: Möbius representation of central capacities for alternatives taking into account imprecise evaluations of alternatives on considered criteria, preferences on importance and interaction of criteria and comparisons between alternatives

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_1	0.31	0.19	0.18	0.19	0.10	0.03	0.00	0.10	-0.10	-0.00
a_3	0.41	0.16	0.25	0.21	0.08	-0.11	0.02	0.08	-0.08	-0.03
a_5	0.29	0.16	0.20	0.22	0.07	0.03	0.04	0.07	-0.11	0.03
a_7	0.32	0.21	0.21	0.19	0.11	-0.04	0.01	0.09	-0.09	-0.01
a_{11}	0.30	0.17	0.16	0.19	0.08	0.10	0.01	0.08	-0.09	-0.00
a_{13}	0.34	0.19	0.21	0.19	0.11	-0.04	0.00	0.09	-0.10	-0.01
a_{15}	0.39	0.17	0.25	0.21	0.09	-0.10	0.02	0.09	-0.09	-0.03
a_{16}	0.32	0.16	0.25	0.21	0.05	-0.04	0.05	0.06	-0.11	0.05
a_{17}	0.29	0.19	0.17	0.19	0.10	0.07	0.00	0.10	-0.10	-0.00

a_{11} and a_1 are weakly preferred to all other alternatives with a frequency of at least 47.04% and 44.09%, respectively (vs the 40.20% of a_7) and both of them are preferred to a_7 more frequently than the viceversa. At the same time, a_9 can be considered surely the worst alternative because all alternatives are weakly preferred to it with a frequency at least equal to the 99.54%.

Computing the Möbius representation of the barycenter of compatible capacities shown in Table 7 and applying the Choquet integral to the average evaluation matrix we get the following ranking of the considered alternatives:

$$a_{17} \succ a_{11} \succ a_1 \succ a_7 \succ a_{13} \succ a_{15} \succ a_5 \succ a_3 \succ a_{16} \succ a_8 \succ a_{18} \succ a_2 \succ a_{14} \succ a_{12} \succ a_4 \succ a_6 \succ a_{10} \succ a_9.$$

5.2 An example with the criteria expressed on different scales

In this section, we deal with a decision making problem in which the evaluations of alternatives on considered criteria are expressed on heterogeneous scales.

From the city-car segment market, we select ten cars evaluated on the basis of the following criteria: price (in Euro), acceleration (0/100 km/h in seconds), maximum speed (in km/h) and consumption (in l/100km) (see Table 8). In this example, we shall suppose that price, acceleration and consumption have a decreasing direction of preference (the lower the evaluation of an alternative on the criterion, the better the alternative is on the considered criterion), while criterion maximum speed has an increasing direction of preference (the higher the evaluation of an alternative on a criterion, the better the alternative is on the considered criterion). Let us notice that, in some cases, criteria

Table 6: Pairwise winning indices taking into account imprecise evaluations of alternatives on considered criteria, preferences on importance and interaction of criteria and comparisons between alternatives

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
a_1	0.00	99.09	92.23	99.70	91.39	99.98	55.80	98.53	100.00	100.00	47.59	99.88	80.36	99.85	85.65	94.32	44.09	99.16
a_2	0.91	0.00	29.52	78.07	14.31	99.20	2.96	37.69	99.99	99.33	0.25	83.54	6.58	77.66	23.08	0.00	0.39	45.69
a_3	7.77	70.48	0.00	93.07	43.69	99.57	8.24	61.99	100.00	99.87	8.77	95.18	18.31	100.00	35.82	53.00	8.02	66.50
a_4	0.30	21.93	6.93	0.00	9.17	86.72	0.23	16.44	100.00	95.37	0.45	51.96	0.97	42.25	6.56	11.77	0.23	21.00
a_5	8.61	85.69	56.31	90.83	0.00	99.99	15.78	65.84	100.00	99.85	2.72	94.92	27.39	94.32	46.07	62.74	5.17	73.95
a_6	0.02	0.80	0.43	13.28	0.01	0.00	0.14	2.68	99.59	62.13	0.00	9.16	0.13	4.64	0.78	0.04	0.00	2.85
a_7	44.20	97.04	91.76	99.77	84.22	99.86	0.00	96.75	100.00	100.00	43.89	99.73	76.56	99.37	84.76	90.05	40.20	96.87
a_8	1.47	62.31	38.01	83.56	34.16	97.32	3.25	0.00	100.00	99.93	2.64	86.11	0.00	79.90	29.48	42.28	1.66	58.17
a_9	0.00	0.01	0.00	0.00	0.00	0.41	0.00	0.00	0.00	0.20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{10}	0.00	0.67	0.13	4.63	0.15	37.87	0.00	0.07	99.80	0.00	0.00	4.19	0.00	3.89	0.28	0.19	0.00	0.19
a_{11}	52.41	99.75	91.23	99.55	97.28	100.00	56.11	97.36	100.00	100.00	0.00	99.94	77.93	99.96	84.83	97.71	47.04	98.97
a_{12}	0.12	16.46	4.82	48.04	5.08	90.84	0.27	13.89	100.00	95.81	0.06	0.00	0.60	39.11	5.38	7.14	0.04	17.50
a_{13}	19.64	93.42	81.69	99.03	72.61	99.87	23.44	100.00	100.00	100.00	22.07	99.40	0.00	99.07	69.73	82.77	18.51	91.79
a_{14}	0.15	22.34	0.00	57.75	5.68	95.36	0.63	20.10	100.00	96.11	0.04	60.89	0.93	0.00	7.73	9.83	0.05	24.04
a_{15}	14.35	76.92	63.57	93.44	53.93	99.22	15.24	70.52	100.00	99.72	15.17	94.62	30.27	92.27	0.00	63.15	14.14	72.99
a_{16}	5.68	100.00	47.00	88.23	37.26	99.96	9.95	57.72	100.00	99.81	2.29	92.86	17.23	90.17	36.85	0.00	4.06	65.09
a_{17}	55.82	99.61	91.98	99.77	94.83	100.00	59.80	98.34	100.00	100.00	52.96	99.96	81.49	99.95	85.86	95.94	0.00	99.46
a_{18}	0.84	54.31	33.50	79.00	26.05	97.15	3.13	41.83	100.00	99.81	1.03	82.50	8.21	75.96	27.01	34.91	0.54	0.00

Table 7: Möbius representation of the barycenter of the compatible capacities taking into account interval evaluations of alternatives on considered criteria, preference information on importance and interaction of criteria and comparisons between alternatives

$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
0.31	0.18	0.19	0.19	0.097	0.034	0.008	0.091	-0.08	-0.014

are non monotonic with respect to the preferences of the DM. This means that one can not state that the criterion has a decreasing or an increasing direction of preference.

Let suppose that the DM supplies the following preference information in terms of importance and interaction of criteria as well as in terms of comparisons between alternatives:

- $\varphi(\{g_1\}) > \varphi(\{g_2\})$, $\varphi(\{g_4\}) > \varphi(\{g_3\})$,
- $\varphi(\{g_3, g_4\}) > 0$, $\varphi(\{g_2, g_3\}) < 0$.
- $a_5 \succ a_1$, $a_7 \succ a_6$, $a_2 \succ a_3$,

As explained in Section 4, at each iteration we sample a common scale, and, if the set of constraints E^{DM} is feasible, then we check if there exists a capacity compatible with these constraints. Let us

Table 8: Evaluation matrix

Cars	Price Euro	Acceleration 0/100 km/h	Max speed km/h	Consumption l/100km
PEUGEOT 208 1.6 8V e-HDi 92 CV Stop&Start 3p. Allure	17,800	10.9	185	3.8
Citroen C3 1.4 HDi 70 Seduction	15,750	13.5	163	3.8
FIAT 500 0.9 TwinAir Turbo Street	15,050	11	173	4
SKODA Fabia 1.2 TDI CR 75 CV 5p. GreenLine	15,260	14.2	172	3.4
LANCIA Ypsilon 5p 1.3 MJT 95 CV 5p. S&S Gold	16,300	11.4	183	3.8
RENAULT Clio 1.5 dCi 90 CV 3p. Dynamique	16,050	11.3	176	4
SEAT Ibiza ST 1.2 TDI CR Ecomotive	15,700	14.6	173	3.4
ALFA ROMEO MiTo 1.3 JTDm 85 CV S&S Progression	17,500	12.9	174	3.5
TOYOTA Yaris 1.5 Hybrid 5p. Lounge	17,800	11.8	165	3.2
VOLKSWAGEN Polo 1.2 TDI 5p. BlueMotion 89g	17,060	13.9	173	3.4

notice that since the DM has provided some preference in terms of comparison between alternatives, the set of constraints E^{DM} will be dependent on the sampled scale.

At the end of all the iterations, we shall get the rank acceptability indices, the Möbius representations of the central capacities for each alternative and the preference matrix shown respectively in Tables 10, 11 and 12 in the Appendix.

In Table 10, we observe that car a_4 is the most preferred by the DM ($b_4^1 = 55.51\%$) followed by a_7, while a_6 is most frequently the least preferred car ($b_6^{10} = 53.04\%$) and a_1, a_2, a_3 and a_6 can never arrive first. Table 11 gives the Möbius representations of the central capacities ranking considered alternatives in the first position at least once, while from Table 12, giving the frequency of the weak preference between pairs of alternatives, we observe that a_4 is weakly preferred to all other alternatives with a frequency at least equal to 67.71%.

Since there are different common compatible scales, we propose the most discriminant common scale, presented in Table 9, to the DM.

Table 9: Evaluations of alternatives on considered criteria expressed on the most discriminating common scale

Alt	Price	Acceleration	Max speed	Consumption
a_1	0.1834	0.7290	0.8208	0.5723
a_2	0.5870	0.4023	0.2107	0.5723
a_3	0.8663	0.6981	0.4427	0.0496
a_4	0.8567	0.1268	0.4234	0.7090
a_5	0.5613	0.5854	0.6979	0.5723
a_6	0.5721	0.6626	0.5906	0.0496
a_7	0.7443	0.0569	0.4427	0.7090
a_8	0.3115	0.4438	0.5717	0.6015
a_9	0.1834	0.5816	0.3944	0.8207
a_{10}	0.4113	0.3501	0.4427	0.7090

After the DM accepts the common scale, we apply SMAA sampling capacities compatible with the preference information provided by the DM, computing the rank acceptability indices, the Möbius representations of the central capacities and the preference matrix displayed, respectively, in Tables 13, 14 and 16 in the Appendix. Applying the Choquet integral with respect to the barycenter of the compatible capacities whose Möbius representation are shown in Table 15, and considering the most discriminating common scale we get the following ranking of the considered alternatives:

$$a_5 \succ a_4 \succ a_7 \succ a_1 \succ a_{10} \succ a_8 \succ a_2 \succ a_9 \succ a_3 \succ a_6.$$

6 Conclusions

In this paper, we have combined the Stochastic Multiobjective Acceptability Analysis (SMAA) to the Choquet integral preference model extending a work already published by the authors [1]. We have proposed to explore the space of the parameters compatible with some preference information provided by the DM using SMAA. In particular, we have considered the DM's preference information not only in terms of relative importance of criteria and interaction between them, but differently from [1], also in terms of pairwise comparison between alternatives and comparisons of intensity of preferences between pairs of alternatives. Moreover, again differently from [1], we have considered also imprecise evaluations of alternatives on the considered criteria.

Finally, we have proposed a methodology to construct the common scale required by the Choquet integral; this is very useful in case the criteria for the decision problem at hand are defined on different scales. Such aspect of the methodology we are proposing constitutes another original contribution with respect to [1]. We have provided some didactic examples in which the proposed methodology has been applied. We envisage the following future developments:

- application of SMAA methodology to some extensions of the classical Choquet integral, e.g. the bipolar Choquet integral [15, 16], the level dependent Choquet integral [19], the robust Choquet integral [21];
- application of the SMAA methodology to the Choquet integral in presence of hierarchy of criteria [2] within the so called multiple criteria hierarchy process [8].

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Appendix

Table 10: Rank acceptability indices sampling simultaneously compatible capacities and scales

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
a_1	0	0.08	0.1	1.61	5.03	18.28	32.05	26.28	4.07	12.5
a_2	0	0.02	1.9	4.12	4.99	14.82	45.23	26.46	2.46	0
a_3	0	0	0	0.57	0.71	1.02	2.98	15.04	52.03	27.65
a_4	55.51	32.37	7.54	3.45	0.99	0.1	0.04	0	0	0
a_5	4.45	2.78	18.38	12.17	23.54	36.88	1.14	0.47	0.19	0
a_6	0	0	0	0.05	0.41	0.89	1.91	10.46	33.24	53.04
a_7	26.15	53.89	11.96	5.46	2.29	0.19	0.06	0	0	0
a_8	4.79	4	20.66	35.32	23.96	6.21	2.88	1.27	0.81	0.1
a_9	5.9	1.72	8.38	9.51	20.36	15.11	9.08	17.92	5.83	6.19
a_{10}	3.2	5.14	31.08	27.74	17.72	6.5	4.63	2.1	1.37	0.52

Table 11: Möbius representations of central capacities sampling simultaneously compatible capacities and scales

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_4	0.06	0.09	0.09	0.17	0.00	0.01	0.48	-0.05	0.08	0.06
a_5	0.09	0.17	0.17	0.16	0.00	0.01	0.38	-0.09	0.02	0.09
a_7	0.06	0.10	0.10	0.18	0.00	0.01	0.48	-0.05	0.05	0.07
a_8	0.06	0.10	0.10	0.17	0.00	0.02	0.41	-0.05	0.11	0.09
a_9	0.05	0.04	0.04	0.28	0.00	0.03	0.34	-0.02	0.17	0.05
a_{10}	0.04	0.07	0.07	0.17	0.00	0.01	0.46	-0.04	0.14	0.08

Table 12: Pairwise winning indices considering a simulation sampling of random capacities and common scales

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
a_1	0	44.25	82.51	0.27	0	85.29	0.32	5.45	34.33	9.63
a_2	55.75	0	100	0.07	4.04	97.45	0.09	9.72	35.01	11.35
a_3	17.49	0	0	0	1.37	65.79	0	2.81	11.38	3.26
a_4	99.73	99.93	100	0	93.99	100	67.71	91.54	93.13	91.47
a_5	100	95.96	98.63	6.01	0	99.45	7.36	34.36	58.28	33.69
a_6	14.71	2.55	34.21	0	0.55	0	0	1.94	9.26	2.58
a_7	99.68	99.91	100	32.29	92.64	100	0	89.46	91.59	89.77
a_8	94.55	90.28	97.19	8.46	65.64	98.06	10.54	0	75.23	48.33
a_9	65.67	64.99	88.62	6.87	41.72	90.74	8.41	24.77	0	21.94
a_{10}	90.37	88.65	96.74	8.53	66.31	97.42	10.23	51.67	78.06	0

Table 13: Rank acceptability indices taking into account evaluations of alternatives on considered criteria expressed on the most discriminating common scale shown in Table 9

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
a_1	0	17.76	8.48	22	10.92	15.65	12.82	9.17	2.42	0.78
a_2	0	2.23	5.39	19.02	12.61	7.36	12.66	39.6	1.13	0
a_3	0	0	0.33	1.53	2.79	3.18	2.93	7.79	69.46	11.99
a_4	32.28	41.1	14.26	6.91	3.2	2	0.25	0	0	0
a_5	65.88	12.06	20.58	1.24	0.24	0	0	0	0	0
a_6	0	0	0	0	0	0.01	0.73	2.05	14.97	82.24
a_7	0.81	21.11	39	15.1	8.06	5.53	5.91	2.38	2.1	0
a_8	0	0.35	4.3	5.79	21.08	28.94	29.92	8.89	0.73	0
a_9	1.03	2.4	5.23	7.01	15.7	9.83	18.64	26.38	8.79	4.99
a_{10}	0	2.99	2.43	21.4	25.4	27.5	16.14	3.74	0.4	0

Table 14: Möbius representation of the central capacities, taking into account evaluations of alternatives on considered criteria expressed on the most discriminating common scale, shown in Table 9

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_4	0.21	0.18	0.15	0.31	-0.01	0.02	0.17	-0.06	-0.03	0.08
a_5	0.15	0.16	0.16	0.18	0.04	0.03	0.14	-0.06	0.07	0.12
a_7	0.03	0.26	0.16	0.30	-0.01	0.07	0.38	-0.11	-0.13	0.04
a_9	0.24	0.15	0.16	0.46	-0.03	0.03	-0.11	-0.07	0.11	0.07

Table 15: Möbius representation of the barycenter of capacities taking into account evaluations of alternatives on the considered criteria expressed on the most discriminant common scale shown in Table 9

$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
0.17	0.17	0.16	0.23	0.02	0.03	0.15	-0.06	0.04	0.10

Table 16: Pairwise winning indices taking into account evaluations of alternatives on the most discriminant common scale shown in Table 9

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
a_1	0	64.01	92.08	20.67	0	97.97	31.4	71.33	77.54	54.86
a_2	35.99	0	100	4.12	0	98.87	11.73	44.58	50.08	35.12
a_3	7.92	0	0	0.82	0	87.97	3.86	8.43	16.46	6.53
a_4	79.33	95.88	99.18	0	33.34	100	97.25	92.89	93.26	94.22
a_5	100	100	100	66.66	0	100	78.61	100	97.19	99.64
a_6	2.03	1.13	12.03	0	0	0	0	0.74	5.36	0.01
a_7	68.6	88.27	96.14	2.75	21.39	100	0	83.14	82.55	83.94
a_8	28.67	55.42	91.57	7.11	0	99.26	16.86	0	64.53	33.65
a_9	22.46	49.92	83.54	6.74	2.81	94.64	17.45	35.47	0	29.4
a_{10}	45.14	64.88	93.47	5.78	0.36	99.99	16.06	66.35	70.6	0